

Event-by-Event Analysis and the Central Limit Theorem

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Abstract

Event-by-event analysis of heavy-ion collision events is an important tool for the study of the QCD phase boundary and formation of a quark-gluon plasma. A universal feature of phase boundaries is the appearance of increased fluctuations of conserved measures as manifested by excess measure variance compared to a reference. In this paper I consider a particular aspect of EbyE analysis emphasizing global-variables variance comparisons and the central limit theorem. I find that the central limit theorem is, in a broader interpretation, a statement about the scale invariance of total variance for a measure distribution, which in turn relates to the scale-dependent symmetry properties of the distribution.. I further generalize this concept to the relationship between the scale dependence of a covariance matrix for all conserved measures defined on a dynamical system and a matrix of covariance integrals defined on two-point measure spaces, which points the way to a detailed description of the symmetry dynamics of a complex measure system. Finally, I relate this generalized description to several recently proposed or completed event-by-event analyses.

1 Introduction

With the advent of high-multiplicity heavy-ion collisions at the CERN SPS and RHIC there is the possibility to extract statistically significant dynamical information from individual collision events. By determining the degree of correlation of the multiparticle final state for individual events within a large ensemble of events one can probe the dynamical history of collisions and the QCD phase boundary. This process is called event-by-event (EbyE) analysis.

A central issue for EbyE analysis is the question whether or not collision events with similar initial conditions experience the same dynamical history, excepting finite-number fluctuations and known hadronic variance sources. If similarly prepared events differ significantly from one another beyond expectation, then analysis must proceed to categorize events into dynamical classes or on a continuum, and to provide a physics interpretation for nontrivial variations.

Based on very general arguments and the observed behavior of normal bulk matter near phase boundaries we expect to observe significant changes in fluctuations near the QCD phase boundary.

The amplitude and character of these changes and their observability given finite collision systems and non-equilibrium collision trajectories are major questions for EbyE physics. Experimentally, the problem calls for extracting all available information from each event by a complete topological analysis in order to obtain maximum sensitivity to dynamical fluctuations.

Various measures can be used to categorize events by their ‘morphology.’ The common element of all measures is the correlation content of the multiparticle final state. Correlation measures can be loosely grouped into *global measures*, some of which have counterparts in thermodynamics, and *scale-local measures*, which explicitly determine the scale dependence of correlations. Given an event-wise measure we also require a corresponding *comparison* measure to enable quantitative event comparisons.

In this paper I emphasize EbyE analysis techniques which relate to variances and second moments. These global statistical measures are directly related to two-point correlation measures defined in various pair spaces. Higher moments are related to general q -point correlations and are more efficiently dealt with by a generalized system of *scale-local* EbyE analysis which will be treated in subsequent papers. The emphasis on variances and two-point correlations in this paper is both timely and provides a reasonably accessible introductory treatment of EbyE analysis.

The paper describes the general types of EbyE analysis, the available system of correlation measures, the system of ensembles and its relationship to topological partition systems, considers maximally symmetric systems and the statistical model as archetypal correlation reference systems, the relationship between fluctuations and correlations, introduces the central limit theorem as an archetype for variance comparison, considers a fluctuation comparison measure (Φ_{p_t}) as a difference between *rms* fluctuation amplitudes, analyzes the structure of Φ_{p_t} as a specific variance comparison measure, introduces the concept of scale-dependent covariance matrices as a generalized measure system, relates scale-dependent covariance to a system of two-point correlation integrals, and considers the novel connections between scale invariance and the central limit theorem which arise from this work.

2 Correlation Measures and EbyE Analysis

Correlation measures provide the basic analytical infrastructure for EbyE analysis. A complete characterization of the correlation content of the multiparticle final state should exhaust all significant information in the data. Comparison of event-wise correlations in a data system with those of a reference system (which may invoke ‘equilibrium’ or an assumption of maximum symmetry) offers a precise means to detect deviations from a model or ensemble average. Significant deviations provide a basis for discovering new physics. One then attempts to understand the physical origin of these deviations, leading to a revised model as part of an oscillatory (but hopefully converging) scientific process.

2.1 EbyE analysis types

We can distinguish two types of event-by-event analysis as represented in Fig. 1. In analysis type A each event is fully characterized for information content and recorded in an ‘event spectrum,’ a space designed to represent differing event morphologies in an efficient manner. This could be as simple as a 1-D space recording the event-wise slope parameter of m_t distributions or the event-wise $\langle p_t \rangle$. Or, it could be a complex scale-dependent representation of correlations in (η, ϕ) distributions for a minijet analysis. Through the intermediary of event-spectrum distributions object and reference event spectra are compared for nonstatistical differences in event morphology. In this analysis we

are usually not interested in the ‘DC’ aspects of correlation common to every event, but only in those ‘AC’ aspects that change from one event to another.

Type A analysis generally requires large acceptance and large event multiplicity. One attempts to characterize all correlation aspects of each collision event over most of phase space, at least implicitly using q -particle correlations, where q may range up to a significant fraction of the event multiplicity. Questions of interest include large-scale variations in collision dynamics (dynamical fluctuations) such as flow fluctuations, baryon stopping fluctuations, jets, departures from chemical and thermal equilibration and chiral symmetry variations. The identity of individual events is preserved for essentially all aspects of this analysis.

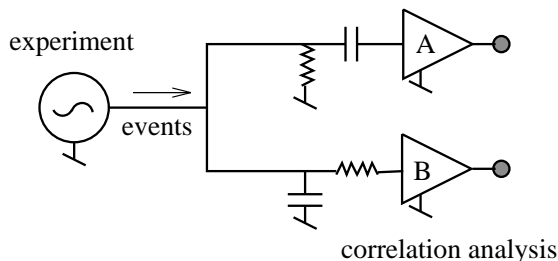


Figure 1: Signal processing analog of EbyE analysis showing AC (A) and DC (B) aspects. Type A analysis emphasizes the differences in correlation content among individual events. Type B analysis emphasizes the correlation content common to an ensemble of events relative to a reference population.

In analysis type B elements of each event are accumulated in an inclusive *sibling* space (*e.g.*, a pion pair space for HBT analysis or an azimuthal space with registered event planes for flow analysis) and compared to a mixed-element space (no siblings by construction, hence minimally correlated) to determine differential inclusive symmetry properties of the event ensemble. Here we are interested in the ‘DC’ aspects of correlation which are true for almost every event, but which may deviate from a model expectation. The identity of individual events is typically discarded in type-B analysis.

If statistical measures (such as moments) are extracted from an event spectrum obtained in a type-A analysis in such a way that event identities are lost then the result is effectively a type-B analysis, even though the statistics were obtained through a type-A analysis. For instance, the moments of a $\langle p_t \rangle$ distribution form the basis for a type-B analysis, although the $\langle p_t \rangle$ distribution itself is the event spectrum for a type-A analysis. The variance of a $\langle p_t \rangle$ distribution contains information which is closely related to that of an inclusive pair-correlation analysis. As we shall see however, this is a complex relationship. In contrast, an analysis in which features of an event spectrum are used to select special event classes is definitely a type-A analysis. Special event classes can then be used as the basis for conventional *inclusive* analyses (*e.g.*, particle ratios or baryon stopping) which would not be feasible on a single-event basis due to limited statistical power but may be essential to interpret the special event class.

For type-B analysis large acceptance and large event multiplicity are still highly desirable. For N-N *vs* A-A comparisons type-B analysis may be the only practical study method due to the low multiplicity of N-N events. Comprehensive analysis of two-point densities over a large acceptance requires *simultaneous* measurement over the entire two-point space to minimize systematic error. As we shall see, in-depth EbyE analysis is often statistics limited and depends on access to the largest possible multiparticle phase space, practical conditions that require the largest possible detector acceptance. EbyE analysis based on variance comparisons as discussed in this paper represents only the most elementary initial approach. More complex extensions to arbitrary q -tuple analysis and scale-local analysis of higher-dimensional measure distributions will extend the initial studies

of simple 1-D distributions described here.

2.2 Correlation measures

The basic tools of EbyE analysis are correlation measures. Each multiparticle event distribution has a certain information content relative to a reference. In principle, all significant event information can be extracted to an efficient system of correlation measures by a data-compression process. If a distribution nears maximum symmetry within constraints the event information may be represented by a very simple measure system. We can divide correlation measures roughly into global measures, which are effectively integrals over some scale interval, and scale-dependent or scale-local measures. A central aspect of EbyE analysis is comparison between object distributions and references. Thus, every correlation measure requires an associated comparison measure. A summary of correlation measures is presented in Table 1.

measure	comparison measure
global measures	
moments	central-limit tests
$W_m(D) = \langle x^m \rangle^{1/m}$	$\Phi_x(D_1, D_2, m)$
model distributions	<i>GoF</i> criteria
$f(x, \alpha)$	$\chi^2\{f(x, \alpha), D\}$
scale-dependent measures	
factorial moments	normed factorial moments
$\mu_{(q)}(D, \delta x)$	$F_q(D_1, D_2, \delta x)$
SCA	
$S_q(D, \delta x)$ entropy	$I_q(D_1, D_2, \delta x)$ information
$d_q(D, \delta x)$ dimension	$\Delta d_q(D_1, D_2, \delta x)$ dim. transp.
$V_q(D, \delta x)$ volume	$\rho_q(D_1, D_2, \delta x)$ density

Table 1: Correlation measures and associated comparison measures. D is a distribution or data set, (D_1, D_2) is an object/reference pair of distributions or data sets, α is a set of model parameters, *GoF* is goodness-of-fit, q is a q -tuple index, δx is a scale (bin size) parameter.

Global measures include distribution moments and model parameters. The respective comparison measures are moment differences (*e.g.*, central-limit tests) and goodness-of-fit (*GoF*) measures (χ^2 or likelihood). The reference may be represented either by a set of moments or by a restricted region in the parameter space of a model distribution. Event-wise information then consists of deviations from the reference represented by moment differences or parameter deviations. As we shall see, global measures are effectively based on scale integrals, although this may not be obvious from typical usage. Integral measures are more appropriate in the case of low statistical power (low event multiplicity) and small correlation content, when only one or a few significant parameters can be extracted from an event.

Scale-dependent measures become more important with increasing event multiplicity and/or correlation amplitude. It is then possible for an event to carry more information, and one can determine the scale and space dependence of correlations. A system of measures based on the scale-dependent correlation integral has been developed in two closely-related manifestations. Factorial moment analysis has a substantial history in multiparticle analysis [1] and is based on the factorial moment as scale-dependent measure and the normalized factorial moment or moment ratio as the comparison measure. Scaled correlation analysis (SCA) is based on a system of generalized entropies [2, 3] extended to scale-local form. The three closely-related topological measures entropy, dimension and volume are indexed by q -tuple number and scale. The corresponding comparison

measures are information, dimension transport and density. In some applications model distributions such as the negative-binomial distribution have been employed in scale-dependent analyses [4, 5]. It is possible to make a connection between scale-dependent model parameters and SCA measures.

3 Ensembles

Many-body dynamics, equilibration and fluctuations – issues central to EbyE analysis – can be studied with an ensemble approach. We encounter ensemble and partition concepts repeatedly in this paper. An ensemble in the sense of statistical mechanics is a set of nominally identical dynamical systems (events) prepared with the same constraints but otherwise uncorrelated. Under Gibbs’ ergodic hypothesis the phase-space distribution of system states of the ensemble at any time should reflect the phase-space history of an individual system state over a long interval. For nearly symmetric systems this hypothesis provides an essential reference.

A generalized ensemble can be formed from an arbitrarily large bounded volume Ω containing a measure system in ‘statistical equilibrium’ (maximally symmetric within global constraints). Ω is invested with a set of conserved global extensive quantities (additive measures) such as total energy \mathcal{E} , total volume \mathcal{V} and total object number \mathcal{N} . This list extends to all quantities conserved by a given dynamical system and is represented below by $\vec{m} = \{m_i\}$. The global volume is partitioned arbitrarily into elementary volumes V_e . Each volume element V_e contains local samples of extensive quantities such as energy E_e and number N_e . Each sample set $(E, N)_e$ provides local estimators $(E/N, E/V, N/V)_e$ for global intensive quantities (T, ϵ, ρ) representing the global constraints $(\mathcal{E}, \mathcal{V}, \mathcal{N})$. The (infinite) set of elements from all possible partitions of Ω I designate the *general ensemble* (GE). This description of a general ensemble adheres closely to the language of Borel measure theory so that topological analysis methods can more easily be connected to the statistical study of dynamical systems. A system of ensembles is illustrated in Fig. 2.

The measure contents of GE partition elements can be represented in a single-point measure space $G_1(\vec{m})$ (cf Secs. 9 and 11). The measure space in Fig. 2 is a subspace $G_1(E, N)$. For a minimally correlated system the GE measure distribution is localized near the main diagonal of $G_1(\vec{m})$. The degree of localization (fluctuation amplitude or variance) depends on the *effective* total number of dynamical objects \mathcal{N} in the space Ω and the fractional volume or scale of typical ensemble elements measured by position along the main diagonal. Note that fluctuation amplitudes decrease (are ‘suppressed’) as this partition scale approaches the total volume \mathcal{V} of the space Ω . The distribution becomes sensitive to the global constraints or boundedness of the dynamical system.

The classical microcanonical, canonical and grand canonical ensembles are conventional subsets of the GE. The grand canonical ensemble (GCE) consists of all partition elements with some fixed volume V (system open to particle and energy exchange). The canonical ensemble (CE) is a special case of the GCE in which object number N is held fixed (system open to energy exchange). The microcanonical ensemble is a further special case in which (E, N, V) are all held fixed (closed statistical system). Ω itself can be seen as an element of a microcanonical ensemble for a larger system. More generally, given an arbitrary measure system defined on a partitioned multidimensional space, one can define a hierarchy of subensembles of successively lower dimensionality corresponding to increasing degrees of constraint on partition elements and measures.

For the purpose of EbyE analysis of nuclear collisions the GCE for an equilibrated system can serve as a minimally correlated (maximally symmetric) reference for the transverse phase space of an event ensemble. One asks whether the stopping process (source of total P_t) for a fixed macroscopic collision geometry is the same for each event, excepting finite-number fluctuations,

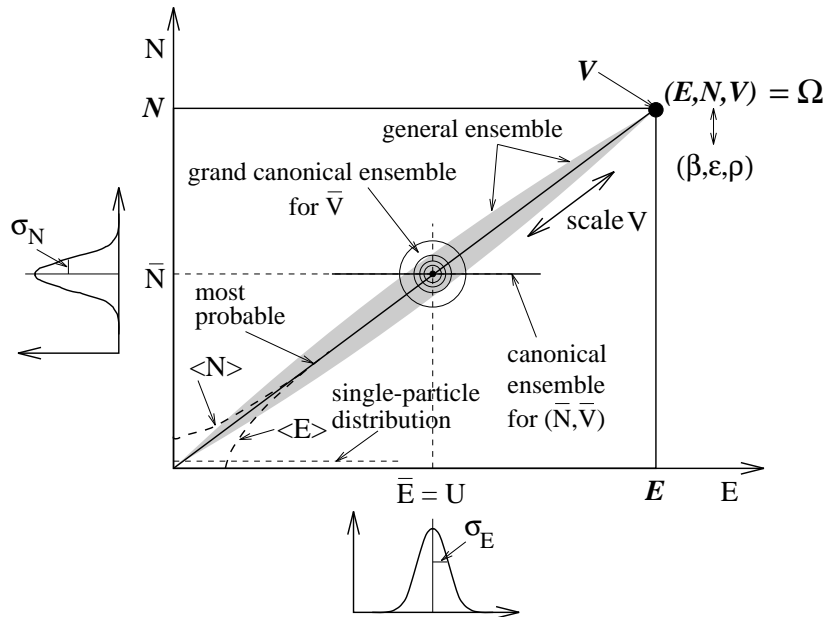


Figure 2: Ensembles: An isolated volume Ω with extensive properties $(\mathcal{E}, \mathcal{N} \dots)$ is defined. Various subsets of the class of all possible partitions of Ω can be identified as ensembles. Contents of partition elements are represented as points in the single-point measure space $G_1(\vec{m})$. Points representing the general ensemble are localized near the main diagonal of this space. The projection of $G_1(\vec{m})$ for the GE onto the $G_1(E, N)$ plane is shown as the shaded region in the figure. The grand canonical ensemble is represented by a slice through this distribution at fixed volume \bar{V} . The result is a localized distribution (concentric circles) centered at (\bar{E}, \bar{N}) with variances σ_E^2 and σ_N^2 .

and whether the ‘stopped’ energy is fully equilibrated among transverse degrees of freedom.

To answer these questions one must compare observed measure distributions for an event ensemble to a suitable reference such as the GCE for an equilibrated system. Multiplicity can be compared to a Poisson, quantum-statistical or hadronic resonance-gas reference. Mean transverse momentum per particle can be compared to an inclusive-spectrum reference, and the linear correlations between these variable pairs can also be compared to an appropriate covariance reference, including effects of hadronic correlations. Agreement between data and the GCE might indicate that in each event N *independent* particle momenta are sampled from a *fixed* inclusive momentum distribution (parent), consistent with independent particle production from an invariant source. Such a comparison test is closely related to the central limit theorem (CLT) and represents the subject of much of this paper.

4 Equilibration and the Statistical Model

By the process of equilibration a many-body system in some arbitrary initial state approaches a state of maximum symmetry. To study this process with sensitivity one needs a well-defined reference and optimized comparison measures. The simplest reference corresponds to an assumption of maximum symmetry within a constraint system, which leads to a ‘statistical-model’ description. Observed deviations from a statistical model may then indicate that an equilibration process from a less

symmetric initial state has terminated before achievement of maximum symmetry or that the true constraint system is incorrectly modeled.

The specific term ‘thermalization’ connotes equilibration of energy in contrast to a more general situation in which various quantities such as charge, color, flavor, spin and baryon number may undergo transport in the approach to maximum symmetry. Equilibration is therefore the preferable term. Equilibrium (zero *net* average transport of conserved quantities) and maximum symmetry are essentially synonymous asymptotic conditions.

4.1 The statistical model

To establish a robust reference system for detecting possibly small deviations which may convey new physics we apply a hypothesis of maximum symmetry (Occam’s Razor, maximum entropy): we adopt the ‘simplest’ hypothesis consistent with global constraints and prior information. The corresponding many-body system is said to be in statistical equilibrium and its mean-value behavior can be described by a *statistical model* (SM).

The statistical model has been applied to simple systems including four-momentum and strangeness as additive measures [6, 7, 8]. A dynamical system in some initial state (possibly partonic) is partitioned at large scale into elements (clusters, hadronic ‘fireballs,’ jets, flow elements) having arbitrary mean momenta. Small-scale system homogeneity is assumed (all clusters have the same intensive properties T and γ_s at hadronization). The large-scale mean momenta of clusters comprise an unthermalized collective flow field which is assumed to play no role in determining particle species abundances. Abundances and ratios are instead assumed to be determined by thermalized energy residing at small scale, with no overlap between these two scale regions.

The partition function for an event composed of individual clusters is a simple product of cluster partition functions with no cross terms, implying linear independence of clusters. Each cluster partition function in turn treats elements (particles) as members of an uncorrelated gas (except for quantum correlations). The treatment is however ‘canonical’ in that some extensive quantities are event-wise conserved, thus introducing a degree of correlation. As we shall see, these assumption of homogeneity, linearity and independence are also preconditions for the central limit theorem.

With the exception of the large-scale flow field the system is assumed to be scale-invariant and maximally symmetric (minimally correlated). Therefore, the system state is determined only by global constraints. Global fitting parameters are T , V and λ_s (quark strangeness suppression factor) or γ_s (hadronic strangeness saturation factor or fugacity). V is the single extensive quantity and should depend on the collision system. T may depend only on the fundamental process of hadronization in elementary systems. An asymptotic (maximum) T value may be observed for small systems in which the hadronic cascade is quickly arrested. In the limit of large system volume and/or particle number this formalism may become consistent with a grand-canonical approach.

The statistical model provides a satisfactory *general* description of multiplicity distributions (revealing primary-hadron or clan correlations) and hadronic abundances/ratios for small and large collision systems over a broad collision-energy interval. An SM treatment of hadron multiplicity distributions for light systems [6] results in two-tiered distributions, with primary hadrons following a nearly Poisson distribution and secondary hadrons distributed according to a negative binomial distribution (NBD) consistent with the number of primary hadrons predicted by the SM. SM treatments of hadron abundances or abundance ratios [7, 8] and strangeness suppression [9] in a variety of collision systems lead to good general agreement with experimental observations in terms of a single universal hadronization temperature $T \approx 180$ MeV and volumes V consistent with

specific collision systems from e^+e^- to the heaviest HI collisions. Although the SM provides a good general summary of hadronic abundances interesting physics may reside in significant disagreements with this model. The SM provides an equilibration reference from which deviations should be the subject of further study.

4.2 Equilibration

A statistical system appears from a macroscopic viewpoint to move persistently toward a state of maximum overall symmetry in an equilibration process. What underlies this apparent teleology is understood to be a probabilistic effect: maximally symmetric macroscopic systems are more likely to be realized in the random evolution of a normal-system microscopic phase space. Symmetry increases with effective phase-space volume (entropy) and *dimensionality*. Increased correlation *decreases* the effective symmetry, volume and dimensionality of a system and corresponds to a less likely macroscopic state. Equilibration then represents a general decrease of system correlation toward a maximally symmetric reference as illustrated in Fig. 3, and can be followed quantitatively with differential correlation measures.

We associate different equilibration types with corresponding transported quantities, such as thermal (momentum) and chemical (flavor) equilibration. The relaxation rates of these processes depend on local cross sections and flux densities. Different equilibration processes may effectively begin and end (freeze out) in different space-time regions of the collision, and with finite transport rates the degree of equilibration is limited by the system space-time extension [10]. An equilibration reference should therefore be defined in terms of finite system size/duration to insure proper treatment of the scale dependence of system symmetries [11]. Concepts such as ‘local’ and ‘global’ equilibration relate to scale dependence, of vacuum symmetries as well as energy/momentum equilibration. Color deconfinement (spatial color symmetry) may be an established condition within cold hadrons. One is more interested in how color and chiral symmetries are modified and *spatially extended* during a HI collision, what is the varying geometry or dimensionality of regions with modified symmetries and how does this dimensionality propagate on scale. Microscopic variations due to small-scale QCD effects are not as interesting as large-scale deviations from a global thermal model that are *unique* to large collision systems. Comparisons between large- and small-system deviations and the scale dependence of correlation within a given system are therefore important.

4.3 Cascades and scale dependence

In conventional treatments of an equilibrated many-body system scale invariance is assumed, and no provision is made to deal with departures from this assumption. If equilibrium is not achieved there has been no quantitative thermodynamic measure system available to deal with deviations. The partition systems usually applied represent very elementary versions of Borel measure theory and treat scaling issues, if at all, with a limited two- or three-tiered structure. Few-stage partition systems in HI collisions represented by jets, strings or primary hadrons combined with a hadronic cascade cannot describe more complex departures from equilibrium expected near a phase boundary.

A classical thermodynamic treatment invokes an extreme scale partition into an ‘atomic’ scale and the system boundary scale. Work can be done on the system boundary ‘adiabatically,’ meaning that energy transferred to the boundary cascades down to the atomic scale within a time small compared to the large-scale energy transfer. The details of the cascade are not described. In thermodynamic language PdV ‘work’ represents large-scale energy transfer. TdS ‘heat’ represents small-scale energy transfer. There is no intermediate possibility. What is ‘adiabatic’ depends on

transfer times at large scale relative to transport times (collision rates) at atomic scale in the thermodynamic medium.

In contrast, scale dependence is an essential ingredient of a comprehensive statistical treatment of many-body dynamics. A cascade process is a hierarchical structure representing the bidirectional propagation of correlation and symmetry on scale. Correlation spontaneously propagates down scale, and equivalently symmetry (dimensionality) propagates upscale in the dominant trend, but with local deviations. In an open system a symmetry increase at larger scale may be implemented ‘opportunistically’ by an accompanying symmetry reduction at smaller scale (*e.g.*, turbulence vortices, biological structures), a process sometimes referred to as ‘spontaneous’ symmetry breaking.

A falling stone collides with a pool of water both at its surface, depositing energy in surface waves, and along a path through its volume, creating a system of vortices. The energy in these large- and medium-scale correlations is transported to smaller scale by a cascade process termed collectively ‘dissipation’ or ‘damping.’ If the dissipation process or cascade is allowed to proceed indefinitely the stopped energy of the stone is finally transported to the smallest scale relevant to the problem, the molecular scale, and the system becomes maximally symmetric or thermalized. If the system is frozen or decoupled before this process is completed residual correlations over some scale interval may be observed. The analog in a HI collision is the interplay among momentum transfers at nuclear, nucleonic and partonic scales and subsequent evolution to final-state hadron momenta at freezeout.

The evolution of an expanding partonic medium into primary hadrons by a coalescence process [12] involves local symmetry reduction (coalescence) accompanying global symmetry increase achieved by the expansion. The decay of these primary hadrons to observed secondaries is a further cascade process, a symmetry increase in which detectable correlations nevertheless persist in the secondary hadron population unless the hadronic cascade can proceed by continued rescattering to a state of maximum symmetry – an equilibrated hadron gas.

The terminology of cascade processes varies with context, as represented by the many terms for localized dynamical objects (quasiparticles) in a nuclear collision: fireballs, primary hadrons, clans, prehadrons, clusters, participants, (wounded) nucleons, resonances, strings, leading partons – resulting in various manifestations of residual correlations in the hadronic final state such as jets, minijets, Cronin effect and NBD. It is desirable to develop a simplified common language which treats these various manifestations of symmetry variation consistently.

4.4 Equilibration tests

Are chemical and thermal equilibrations achieved during a HI collision? Is the final state of an N-N, p-A or A-A collision the result of a completed equilibration process? Or are correlation remnants from the early stages of the collision process still detectable in the multiparticle final state, and do they indicate evidence for QGP formation? Methods are required to distinguish between equilibrium and nonequilibrium models. These questions cannot be resolved by inclusive measurements. To detect and study possible *departures* from equilibrium or maximum symmetry we require a differential correlation analysis system and a reference system analogous to the statistical model but applicable to *fluctuation* analysis. This paper focuses on measurement and interpretation of *variance differences* as one component of a global EbyE fluctuation analysis.

If remnants of a cascade structure are still present after hadronic freezeout we may observe the corresponding residual correlations as excess or deficient (co)variance (fluctuations) of global measures. An elementary equilibration test might therefore consist of comparing the variances of

event-wise measure-distribution statistics to a reference system. A nonequilibrated system might exhibit variances differing from finite-number statistics and elementary hadronic processes.

Gaździcki, Leonidov and Roland (GLR) [13] considered whether p_t production in HI collisions is consistent with a fully equilibrated system or retains evidence of initial-state or pre-equilibrium scattering of projectile nucleons (ISS, Cronin effect). The initial state scattering (ISS) model predicts increased p_t fluctuations because p_t is generated by a *hierarchical process* in which transverse random walk of projectile-nucleon p_t (multiple scattering) is followed by thermal generation of hadrons about the resulting primary p_t distribution. This relationship between hierarchical structure and increased correlation is observed in studies of NBD multiplicity distributions and clans in hadron production in small collision systems [6]. ISS should lead to a correlated p_t population and corresponding fluctuation excess. These fluctuations should increase in amplitude with the number of possible nucleon interactions (thus with nuclear size).

Both equilibrium-hydrodynamic and ISS models provide reasonable descriptions of inclusive p_t distributions. To further distinguish between the two alternatives GLR proposed to compare p_t *fluctuations* predicted by the ISS model and a hydro model with experimental data. The Φ_{p_t} measure which they employed is said to provide a quantitative basis for comparing fluctuations in HI collisions relative to elementary interactions [14].

An additional source of p_t fluctuations according to GLR is the observed $N-\langle p_t \rangle$ correlations in N-N collisions [15], some degree of which may survive an A-A collision considered as a superposition of elementary N-N events. According to GLR surviving $N-\langle p_t \rangle$ correlations would lead to departures from independent-particle emission detectable with an appropriate measure such as Φ_{p_t} applied in a systematic study of $N-\langle p_t \rangle$ correlations in N-N, p-A and A-A. They also suggest that a substantial change in the nuclear medium in A-A collisions could cause a different relationship among elementary N-N collision subevents as indicated by a significant change in differential variance. They conclude that Φ_{p_t} should be sensitive both to $N-\langle p_t \rangle$ correlations and to ISS.

5 Fluctuations and Correlations

Fluctuations and correlations are complementary descriptive terms reflecting the degree of structure in a measure distribution. The term fluctuations may carry a connotation of time dependence, but the term more generally refers to the *variability* of a measure distribution, whether temporal or spatial. Correlation refers to the *localization* or size reduction of a measure distribution on an embedding space. Increased localization leads to increased variability of the measure, possibly interpreted as fluctuations. The terms are sometimes used interchangeably.

Although fluctuations and correlations can be characterized by a complete system of measure distribution moments this paper emphasizes second moments or variances. The primary variance source in a typical ‘many-body’ system is the finite density of measure-carrying objects (degrees of freedom or *DoF*). This concept emerges when a measure system accumulates significant correlation over restricted scale intervals (correlation onset \rightarrow ‘particles’), and only a small degree of correlation over intervening scale intervals (see Sec. 10.2). As a simplification one then focuses on the correlations *among DoF* at large scale, and ignores their internal structure at smaller scale.

A reference density of independent *DoF* corresponds to an expected or basal variance in any physically observable quantity. If the *effective* density of *DoF* changes from a reference value observed fluctuations may depart from expectation. A dramatic change in the number of *DoF* can happen in a phase transition, in which scale-localized correlations are created or destroyed by a change in system constraints.

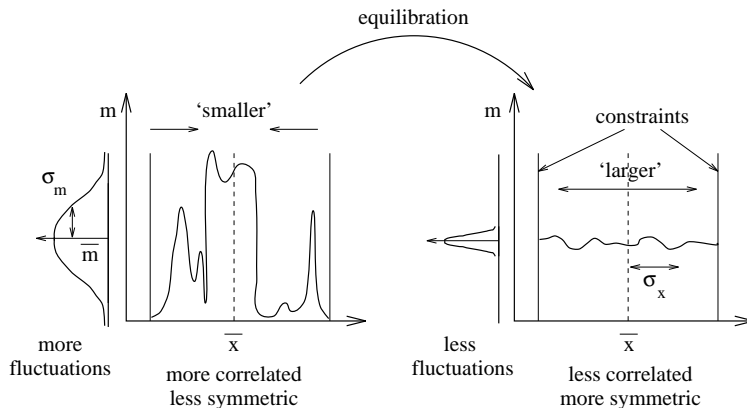


Figure 3: Variation of fluctuations, correlations and symmetry with advancing equilibration. Decreasing correlation corresponds to increase of effective system size (entropy) within constraints, increased symmetry and decreased measure variation (fluctuations). Thus, determination of fluctuations and/or correlations of measure distributions informs us about the equilibration state of a dynamical system.

Changes in fluctuation amplitude can also happen if correlations among *DoF* change. Positive correlation reduces the effective number of *DoF* leading to excess variance beyond expectations. The opposite is true of anticorrelation. One can say equivalently that the entropy associated with a class of *DoF* is reduced through a coupling process, leading to a degree of freezeout of the corresponding *DoF* from an equipartition process. Changes in fluctuation amplitudes or variances can thus be seen as indicators of increased or decreased *correlation among DoF* or as a change in the *density of DoF*, and can be used to probe the underlying Lagrangian for a many-body system or the mechanism of a dynamical process.

Some correlations in a dynamical system are manifestations of incomplete equilibration, the surviving remnants of an incomplete cascade following initial large-scale departures from symmetry. Some correlations may be due to quantum statistics or other steady-state multibody interactions. If excess correlation has been detected beyond that attributable to known sources the problem then lies in identifying the process that led to this correlation and/or what reduced symmetry initiated it. From an EbyE viewpoint deviations from *expected* fluctuation amplitudes (either more or less) which may signal unexpected physics or differences from a reference model are of primary interest.

A central problem for HI physics is the nature of the QCD phase boundary. The location of the boundary can be determined by inclusive measurements [10], but the detailed characteristics of the boundary are more directly probed by EbyE analysis [16]. Fundamental to EbyE QCD studies is the quantitative measurement of changes in QCD vacuum symmetries as energy and baryon-number densities are varied. The most accessible probe of the QCD vacuum may be the multiparticle final state of an URHI collision. Near the phase boundary event-by-event dynamical fluctuations should provide essential information on vacuum symmetry dynamics.

In the partonic regime the highly structured nonabelian gluon field plays a substantial role [17]. At RHIC energies and above the incidence of parton interactions with high momentum transfer should lead to jet formation and a large minijet background in A-A collisions [18], resulting in a highly correlated multiparticle final state at high p_t due to hard processes.

At a phase boundary the density of momentum-carrying objects or degrees of freedom (*DoF*) changes rapidly with certain state variables (*e.g.*, temperature and net baryon density). This means that in the neighborhood of the boundary a bulk-matter system undergoes a sharp change

in pressure and energy density. Fluctuations in extensive quantities exceed those expected from finite-number statistics by an amount bounded by (but not determined by) a corresponding linear response coefficient (*e.g.*, heat capacity) [11]. To determine the detailed properties of the phase boundary it is therefore important to search for and measure nonstatistical fluctuations in dynamical quantities.

However, there are important differences between a phase boundary in bulk matter and *manifestations* of a phase boundary traversed by a small system during a collision, which manifestations may be *barely detectable* in the multiparticle final state [11]. Phase-boundary manifestations compete for observability with details of the collision dynamics, the space-time finiteness of the collision system and the varying degrees of equilibration in different space-time regions of the collision. Potentially observable effects can be attenuated by final-state hadronic interactions which move the system further toward equilibrium, with consequent loss of information, and introduce distracting hadronic correlations which are not of primary interest. There is a critical need therefore to develop a sensitive, complete and interpretable system of correlation measures and comparison measures for heavy-ion collisions.

6 The Central Limit Theorem

I begin a discussion of variance comparison measures with an elementary treatment of the central limit theorem (CLT). In later sections we will develop a more complete view of the CLT and variance comparisons applicable to EbyE analysis. The central limit theorem, first stated by deMoivre and proved under general assumptions by Markov, plays an important role in many statistical applications. It is represented schematically in Fig. 4. The following is an elementary statement of the CLT as typically encountered in a statistical context.

Given a *fixed* arbitrary parent distribution with mean μ and variance σ^2 , and given an ensemble of events, each composed of N -fold *independent* samples from the fixed parent:

- The distribution of event means approaches a gaussian distribution with mean μ and variance σ^2/N .
- More generally, the m^{th} moment about the mean of the event-mean distribution is reduced from the m^{th} moment about the mean of the parent distribution by a factor $1/N^{m-1}$.

The more general (m^{th} moment) aspect of the theorem implies rapid asymptotic approach to a gaussian distribution (itself defined by the requirement that all moments beyond the second are zero) as $N \rightarrow \infty$.

Applied to the problem of event-wise momentum fluctuations in HI collisions as an example the CLT can be formulated in the following way. If events formed from N *independent* p_t samples with total momentum P_t (an ensemble of final-state multiparticle distributions resulting from nominally identical collision events) are drawn from a *static* parent distribution (estimated by the inclusive p_t distribution) the distribution of event means $\langle p_t \rangle = P_t/N$ approximates a gaussian distribution with mean approaching the parent mean \bar{p}_t and variance $\sigma_{\langle p_t \rangle}^2$ related to the parent variance $\sigma_{p_t}^2$ by

$$N\sigma_{\langle p_t \rangle}^2 = \sigma_{p_t}^2 \tag{1}$$

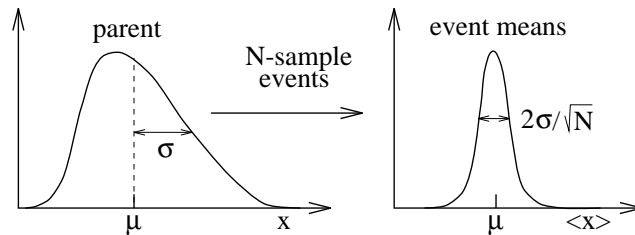


Figure 4: The central limit theorem is illustrated. For an arbitrary parent distribution (left) the distribution of N -fold means approaches with increasing N a gaussian whose width is simply related to the parent width.

Any departures from the CLT are due to 1) (anti)correlated samples and/or 2) variation of the parent during the sample history. It is case 2) that we prefer to identify with ‘dynamical fluctuations,’ although this is not a general convention. The term ‘dynamical fluctuations’ has also been taken to represent all fluctuations beyond finite-number statistics [19]. Net positive correlation implies that $N_{eff} < N$, that is, the *effective* sample is less than N . The opposite is true for anticorrelated samples. It is important to note that the combined presence of different correlation types and dynamical parent variations may nevertheless result in variance behavior *apparently consistent* with the CLT. This ambiguity will be explored below.

7 The Φ_{p_t} Measure

To facilitate the quantitative study of fluctuations and correlations in HI collisions Gaździcki and Mrówczyński (GM) [14] developed a differential correlation measure Φ_{p_t} to study equilibration in HI collisions. This measure and its experimental applications have stimulated much theoretical activity in EbyE analysis recently [20]. GM cite $N-\langle p_t \rangle$ correlations in N-N collisions [15] as a possible means to track the equilibration process in heavier collision systems. For the case which prompted the development of the measure, if A-A collision events are indeed linear superpositions of independent N-N subevents they suggest that a suitably defined fluctuation measure for A-A events should have the same value as for N-N events taken singly if linearity prevails.

According to GM the Φ_{p_t} width comparison measure which they define satisfies two important conditions: 1) linearity - superposing independent and equivalent subsystems to form a composite system results in the same measure value for subsystems and composite; and 2) zero value for independence - if the atomic elements of the subsystems (and thus the composite) are themselves statistically independent the measure is zero. This measure has been proposed as a measure of EbyE p_t fluctuations. It is also proposed as a general measure to detect event-wise dynamical fluctuations in energy and possibly other kinematic or chemical quantities [21].

Rather than comparing variances directly Φ_{p_t} is defined in terms of *rms* quantities with the basic structure

$$\Phi_{p_t} = \sqrt{N} \sigma_{\langle p_t \rangle} - \sigma_{p_t} \quad (2)$$

which is zero if the event ensemble is consistent with the CLT. Following GM one can define an

event-wise variation about the mean

$$\begin{aligned} Z_e &= \sum_{i=1}^{N_e} (x_i - \bar{x}) \\ &= N_e \cdot (\langle x \rangle_e - \bar{x}) \end{aligned} \quad (3)$$

and then define the *rms* comparison measure

$$\Phi_x = \sqrt{Z^2/N} - \sigma_x. \quad (4)$$

This measure satisfies both of conditions 1) and 2) above. It is zero if the event is composed of uncorrelated objects (elements with no resolved structure). The requirement 1), 2) and Eq. (4) with $\Phi_x = 0$ are obviously equivalent to the central limit theorem. Given its fundamental importance it is not surprising that the CLT is frequently rediscovered. GM apply this general comparison measure to single-particle p_t distributions as Φ_{p_t} . To specify Φ_{p_t} one replaces x by p_t in the preceding.

In a Φ_{p_t} analysis of NA49 Pb-Pb data the conclusion was tentatively formed [22, 23, 24] that ISS and $N-\langle p_t \rangle$ fluctuations are substantially reduced, either by collision dynamics and medium effects or by final-state rescattering, to a point below statistical significance and certainly below values observed in p-A and N-N systems respectively. In a further study [25] involving detailed simulations of detector response (two-track resolution) and quantum statistical contributions to Φ_{p_t} a model study was conducted which placed an upper limit on dynamical (parent) fluctuations at 1% of the inclusive p_t distribution *rms* variation.

8 The Structure of Φ_{p_t}

Application of the CLT in the form of Φ_{p_t} to transverse momentum fluctuations has some unanticipated aspects. As suggested above, there is a number of bipolar and positive-definite contributions to Φ_{p_t} . If the measure is ‘zero’ one does not know whether correlations are really absent or instead somehow present in equal magnitude but opposite sign from separate sources, or whether bipolar effects from a single mechanism distributed over a pair-momentum space make no *net* contribution to this integral measure. Even if Φ_{p_t} is nonzero these ambiguities remain. These issues are not confronted in an elementary statement of the CLT as presented above. To better interpret Φ_{p_t} and to formulate improved fluctuation measures we examine the structure of Φ_{p_t} in detail. An important complicating element is the presence of event-wise multiplicity variations, whereas our simple statement of the CLT assumed a fixed sample number N . One must finally accommodate the $N-\langle p_t \rangle$ correlations which originally motivated the development of Φ_{p_t} .

Examining the algebraic structure of Φ_{p_t} in more detail we will find that this measure is only part of a more complex scale-dependent covariance measure system. We will obtain a more comprehensive replacement for the elementary CLT and a generalization of the Φ_{p_t} comparison measure. This fuller picture will enable us to assess possible physical sources of deviation from the CLT and how they are manifest in Φ_{p_t} and associated measures.

Because of potential correlations among statistical quantities and the need for precision in obtaining small differential quantities from large numbers with great statistical power I summarize the essential elementary quantities. $\sum_e = \mathcal{M}$ is a sum over the event ensemble. $\sum_i = N_e$ is a sum over elements of event e . $\sum_{e,i} / \sum_e = \bar{N}$ is the mean event multiplicity. $\sum_i x_i = X_e$ is an event-wise total measure. $\sum_i x_i / \sum_i = X_e / N_e = \langle x \rangle_e$ is an event-wise mean. $\sum_{e,i} x_i / \sum_{e,i} = \bar{x}$

is an ensemble (inclusive) mean over all events. $\sum_e \langle x \rangle_e / \sum_e = \overline{\langle x \rangle}$ is an ensemble-averaged event-wise mean. In general, $\overline{\langle x \rangle} \neq \bar{x}$ because of possible correlations between N_e and $\langle x \rangle_e$. In fact,

$$\overline{N(\overline{\langle x \rangle} - \bar{x})} = -r(\langle x \rangle, N) \sigma_{\langle x \rangle N} = \sigma_{\langle x \rangle N}^2 \quad (5)$$

where the linear correlation coefficient $r(x, y)$ is defined below, and $\sigma_{\langle x \rangle N}^2$ is a covariance.

8.1 Variances

This paper emphasizes global fluctuation measures based on variance. The comparison measure Φ_{pt} is based on an *rms* difference. We now extend the elementary CLT to accomodate event-wise multiplicity variation or variation of the sample number N . We consider the interplay among event-wise total momentum P_e , particle momentum p_i , event-wise mean momentum $\langle p \rangle_e$ and event multiplicity N_e in the spaces $G_1(P, N)$ or $G_1(P/N, N)$. To simplify the notation I omit subscript t and indices e and i wherever confusion will not result.

A differential fluctuation measure based on variances and the CLT can be expressed as

$$\Phi_p = \sqrt{(P - \hat{P})^2 / N} - \sqrt{(p - \bar{p})^2} \quad (6)$$

There are two possible ways to expand the variance $(P - \hat{P})^2$ if the number N of samples or particles fluctuates from event to event, depending on how one defines the mean value \hat{P} . In the first case $\hat{P} = \overline{N\bar{p}}$ and I denote the variance by σ_P^2 . In the second case the mean value $\hat{P} = N_e \bar{p}$ contains the event-wise multiplicity. The variance σ_Φ^2 in the second case is the same $\overline{Z_e^2}$ appearing in the original definition of Φ_{pt} . To simplify I replace $(p - \bar{p})^2$ with σ_p^2 and $\overline{Z_e^2}$ with σ_Φ^2 .

The variance of total momentum σ_P^2 is given by

$$\begin{aligned} \sigma_P^2 &\equiv \overline{(P - \overline{N \cdot \bar{p}})^2} \\ &= \overline{\sum_{i \neq j} p_i p_j} - \overline{N(N-1)} \bar{p}^2 + \overline{N} \sigma_p^2 \end{aligned} \quad (7)$$

The variance used to define the Φ_{pt} measure is given by

$$\begin{aligned} \sigma_\Phi^2 &\equiv \overline{(P - N_e \cdot \bar{p})^2} \\ &= \overline{\sum_{i \neq j} p_i p_j} - \overline{N(N-1)} \bar{p}^2 - 2\bar{p} \cdot \{\overline{NP} - \overline{N^2} \cdot \bar{p}\} + \overline{N} \sigma_p^2 \\ &= \sigma_P^2 - 2\bar{p} \cdot \{\overline{NP} - \overline{N} \cdot \bar{p}\} + \bar{p}^2 \sigma_N^2 \end{aligned} \quad (8)$$

This algebra preserves any N - p_t correlations. No assumptions are made about the statistical interdependence of N and p_t .

8.2 Decomposition of Φ_{pt}

We analyze Φ_{pt} as suggested by the structure of σ_Φ^2 . Recalling that $\Phi_{pt} = \sqrt{\sigma_\Phi^2 / N} - \sigma_{pt}$ we write

$$\sigma_\Phi^2 = A + B + \overline{N} \sigma_p^2 \quad (9)$$

where

$$\begin{aligned} A &= \sum_{i \neq j} p_i p_j - \overline{N(N-1)} \cdot \bar{p}^2 \\ &= \sigma_P^2 - \bar{N} \sigma_p^2 - \bar{p}^2 \sigma_N^2, \end{aligned} \quad (10)$$

$$\begin{aligned} B &= -2\bar{p} \cdot \{\overline{N^2 \langle p \rangle} - \overline{N^2} \cdot \overline{\langle p \rangle}\} \\ &= -2\bar{p} \cdot r(\langle p \rangle, N^2) \cdot \sigma_{\langle p \rangle} \sigma_{N^2} \\ &= -\sigma_P^2 + \sigma_\Phi^2 + \bar{p}^2 \sigma_N^2 \end{aligned} \quad (11)$$

and express Φ_{p_t} as

$$2\sigma_{p_t} \cdot \bar{N} \Phi_{p_t} \approx A + B \quad (12)$$

This decomposition is nontrivial because of the unique roles played by the resulting A and B terms. A is essentially a weighted integral of net two-point correlation, and B is a measure of N - $\langle p_t \rangle$ correlations. Each term is zero under conditions that satisfy the CLT. This decomposition can help to disentangle various physical contributions to Φ_{p_t} . We now examine A and B in more detail.

8.3 Two-point correlations

We provisionally interpret term A as the contribution of two-point correlations to Φ_{p_t} . In the continuum limit this term can be written

$$\begin{aligned} A &= \overline{\sum_{i \neq j} p_i p_j} - \overline{N(N-1)} \cdot \bar{p}^2 \\ &\approx \int p_1 p_2 \cdot \{\rho_2(p_1, p_2) - \rho_1(p_1) \rho_1(p_2)\} dp_1 dp_2 \end{aligned} \quad (13)$$

The quantity in curly brackets in the integrand is the connected two-point correlation function. It is zero in the absence of *net* two-point correlations. The correlation function used in HBT analysis is closely related. The integral is weighted by the prefactor $p_1 p_2$, which leads to an undesirable emphasis on the large- p region. Contributions to A include, but are not limited to, quantum statistics, Coulomb effects, resonance decays, instrumental effects and ‘dynamical’ fluctuations.

It is more useful to study the two-point correlations in Eq. (13) directly rather than in terms of an integral weighted by the factor $p_1 p_2$. One can switch to a mixed-pairs reference

$$\rho_1(p_1) \rho_1(p_2) \rightarrow \rho_{mixed}(p_1, p_2),$$

and compare the two-point densities on (p_1, p_2) for sibling pairs and mixed pairs. A then becomes

$$A \approx \int p_1 p_2 \cdot \{\rho_{sib} - \rho_{mixed}\} dp_1 dp_2 \quad (14)$$

We can plot the ratio of density distributions ρ_{sib}/ρ_{mixed} for sibling and mixed pairs, and also project the ratio onto momentum difference q and mean momentum k by analogy with standard HBT analysis.

To minimize instrumental effects and to reveal possibly different correlation mechanism for different pair types we plot the pair density ratio ρ_{sib}/ρ_{mixed} in Fig. 5 [26] separately for two combinations of like-sign pairs, one of unlike-sign pairs and one for all pairs taken together. One

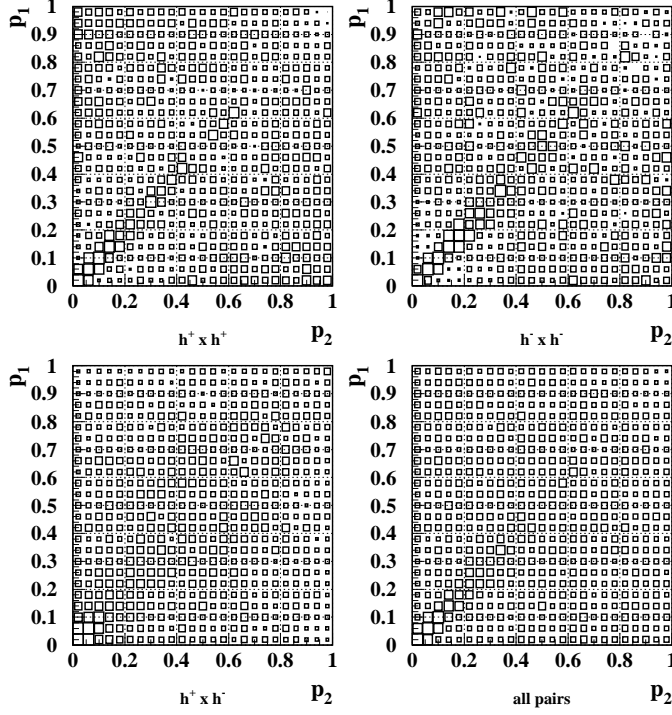


Figure 5: Distribution of sibling to mixed pair-density ratio on $p_t \otimes p_t$ space for 158 GeV/nucleon Pb-Pb collisions. Density-ratio variation (denoted by box size) is within the interval 1 ± 0.003 . The histograms are symmetric about the diagonal by construction. The pair momentum difference is q and the momentum sum is k . The variables p_i are transformed from $m_t - m_0$ in such a way as to make the statistical error approximately uniform over the two-point space.

observes a prominent ridge along the main diagonal ($q = 0$) for the like-signed pairs corresponding to quantum correlations and a peak in the unlike-signed pairs ($k \approx 0$) due to the Coulomb interaction. There is also a very broad *anticorrelation* at large q in the like-signed pair distributions and a broad correlation at large q in the unlike-signed pair distribution. These last results are unexpected and illustrate the difficulty of interpreting Φ_{p_t} in a simple manner. A separate treatment of positive, negative and unlike-signed pairs is generally necessary in order to extract complete information. Different physics may affect each combination.

Evidence for dynamical fluctuations in the parent distribution should be common to all pair types (like and unlike pairs), providing an additional signature of true dynamical fluctuations. Because Φ_{p_t} is an *integral* correlation measure there could be major differences between the ρ_{sib} and ρ_{mixed} density distributions which nevertheless remain undetected using Φ_{p_t} . Scale-local measures and direct analysis of individual pair spaces should provide a much more sensitive approach to the search for dynamical fluctuations.

8.4 N - $\langle p_t \rangle$ correlations

Term B is proportional to a covariance or linear correlation coefficient.

$$\begin{aligned}
 B &= -2\bar{p} \cdot \{ \overline{N^2 \langle p \rangle} - \overline{N^2} \cdot \overline{\langle p \rangle} \} \\
 &\equiv -2\bar{p} r(\langle p \rangle, N^2) \overline{N} \sigma_{\langle p \rangle} \sigma_N \\
 &= -2\bar{p} \sigma_{\langle p \rangle N^2}^2
 \end{aligned} \tag{15}$$

where the linear correlation coefficient (*lcc*) for two variables x and y is defined by

$$r(x, y) = \frac{\overline{(x - \bar{x})(y - \bar{y})}}{\sigma_x \cdot \sigma_y}$$

$$= \frac{(\overline{xy} - \bar{x} \cdot \bar{y})}{\sigma_x \cdot \sigma_y} \quad (16)$$

and the numerator is the covariance σ_{xy}^2 .

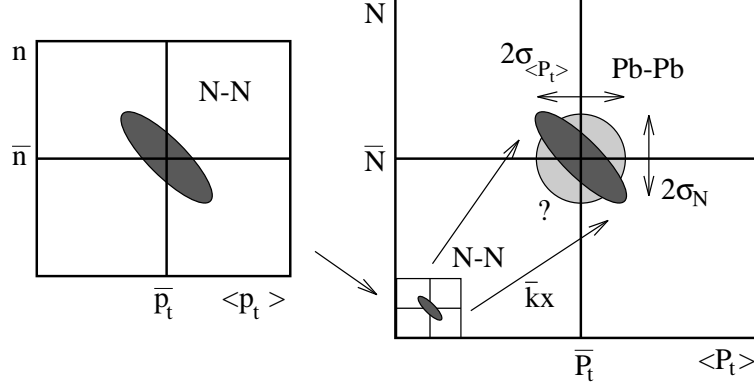


Figure 6: Illustration of N - $\langle p_t \rangle$ correlations in light systems (left) and HI collisions (right). N - $\langle p_t \rangle$ correlations observed in light systems (N-N) may be used as a probe in HI collisions to study equilibration. The question then becomes to what extent N - $\langle p_t \rangle$ correlations survive the equilibration process; how are conservation laws expressed differently in HI collisions than in N-N collisions?

Note that the lcc also has properties 1) and 2) used in the derivation of Φ_{p_t} . It is 1) invariant under linear composition of independent and equivalent subevents and 2) zero in the absence of correlation of event elements. To demonstrate the linearity of this measure consider Pb-Pb collisions as linear superpositions of N-N collisions. If k_e N-N subevents are independently superposed in Pb-Pb event e , $\langle P_t \rangle_e$ is the event-wise mean p_t , $\langle p_t \rangle_j$ is the mean p_t for the j^{th} N-N subevent and n_j is its multiplicity then:

$$\begin{aligned} r(\langle P \rangle, N)_{Pb-Pb} &= \frac{(\langle P_t \rangle - \bar{P}_t) \cdot (N - \bar{N})}{\sigma_{\langle P_t \rangle} \cdot \sigma_N} \\ &= \frac{\sum_{i,j=1}^k \langle p_t \rangle_i \cdot n_j - \bar{P}_t \cdot \bar{N}}{\sigma_{\langle P_t \rangle} \cdot \sigma_N} \\ &= \frac{\bar{k} \cdot \langle p_t \rangle \cdot \bar{n} + \bar{k}(\bar{k} - 1) \cdot \bar{p}_t \bar{n} - \bar{k}^2 \cdot \bar{p}_t \bar{n}}{\bar{k} \cdot \sigma_{\langle p_t \rangle} \cdot \sigma_n} \\ &= r(\langle p \rangle, n)_{N-N} \end{aligned}$$

Thus, the lcc also offers a significant opportunity to probe 1) momentum conservation effects, 2) equilibration processes and 3) independence of particle emission.

The lcc is linear under composition of partition elements (scale invariant) because it is a ratio of total covariance matrix elements – $\sigma_{xy}^2 / \sigma_x \sigma_y$. A scale-dependent factor M cancels in this ratio. If the system is scale invariant (satisfying CLT conditions) then this lcc ratio is invariant also. However, if the CLT is not satisfied this composite measure is not optimal for understanding correlation sources, because it mixes the scale dependence of variances and covariances. It is better to look at $\sigma_{x,y}^2$, σ_x^2 and σ_y^2 or their total-covariance counterparts separately. The intuitively motivated

linearity condition for Φ_{p_t} will become a generalized scale-invariance test of total-covariance matrix elements (cf Sec. 10.2). The question of the evolution of N-N $N-\langle p_t \rangle$ correlations in Pb-Pb collisions which was raised in [13] and [14] is characterized in Fig. 6.

9 A Vector Model of Variance

We find that the variance comparison measure Φ_{p_t} defined in the spirit of the CLT apparently has two algebraic components, A being the integral of net two-point correlation and B containing the lcc for $N-\langle p_t \rangle$ correlations. In summary

$$\begin{aligned} \sigma_\Phi^2 - \overline{N}\sigma_p^2 &= A + B \\ &\approx 2\overline{N}\sigma_p\Phi_{p_t} \end{aligned} \quad (17)$$

We require a clearer picture (a geometrical model) of how the quantities σ_P^2 , σ_Φ^2 and σ_N^2 are related to one another and to the CLT. This problem involving measures P , N and P/N is a particular instance of the generic case ($m_1, m_2, < m > \equiv m_1/m_2$).

I now develop a vector representation that compactly displays the algebraic relationships among the variances. In a later section I adopt a broader description based on the covariance matrix. To simplify notation I use a system of commensurate variables ($P, \overline{p}N, \overline{N}\langle p \rangle$), based on the original ($P, N, \langle p \rangle$) but all having mean value \overline{P} and respective *rms* widths ($\overline{p}\sigma_N, \sigma_P, \overline{N}\sigma_{\langle p \rangle} \equiv \sigma_\Phi$). A sketch of typical 2-D density distributions on these variables with corresponding variances is shown in Fig. 7. For discussion I label these single-point measure spaces G_P and G_Φ . The distributions on G_P and G_Φ are simply related algebraically. For events consisting of independent samples the covariance in G_Φ is small and that in G_P is therefore maximal. As represented by the dashed ellipses in Fig. 7 the opposite could also be true in principle, or any intermediate case could occur. This descriptive problem underlies the Φ_{p_t} -variable analysis. I now develop a vector model of variance which can be used to relate the variances and covariances in these spaces.

9.1 A vector covariance representation

Because of their simplicity the density distributions typically encountered in spaces G_P and G_Φ for an approximately homogeneous system can be well described in terms of their first and second moments (gaussian fluctuation model), with the addition of a covariance element describing the correlation between fluctuations on two degrees of freedom. The covariance matrix in either space can be written formally as a direct product of column/row matrices whose elements are themselves vector representations of variance defined in a pseudospace. For example, we can define a generic covariance matrix

$$\begin{aligned} \mathbf{K}_{xy} &= \begin{pmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{xy}^2 & \sigma_y^2 \end{pmatrix} \\ &= \begin{pmatrix} \vec{\sigma}_x \cdot \vec{\sigma}_x & \vec{\sigma}_x \cdot \vec{\sigma}_y \\ \vec{\sigma}_x \cdot \vec{\sigma}_y & \vec{\sigma}_y \cdot \vec{\sigma}_y \end{pmatrix} \\ &= \begin{pmatrix} \vec{\sigma}_x \\ \vec{\sigma}_y \end{pmatrix} \cdot \begin{pmatrix} \vec{\sigma}_x & \vec{\sigma}_y \end{pmatrix} \end{aligned} \quad (18)$$

where the dot product $\vec{\sigma}_x \cdot \vec{\sigma}_y$ is defined by

$$\vec{\sigma}_x \cdot \vec{\sigma}_y = \sigma_x \sigma_y \cos\theta \quad (19)$$

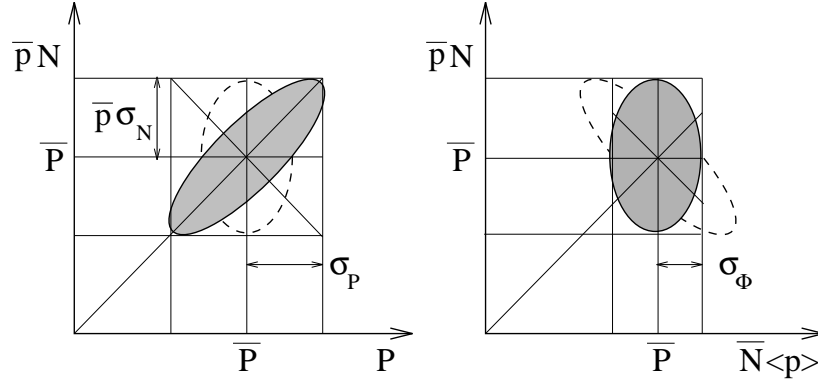


Figure 7: The measure space on the left is $G_P = G_1(p, n)$ and on the right is $G_\Phi = G_1(p/n, n)$. These spaces serve as archetypes for any related set of extensive and intensive (ratio) measures or thermodynamic state variables. The solid ellipses illustrate the case of independent sampling at small scale followed by composition to events, whereas the dashed ellipses illustrate independent sampling at large scale followed by partition to N objects. The major axes of the ellipses include variations of the effective event volume, while the minor axes are determined primarily by statistical variance.

and $\cos\theta \equiv r(x, y)$ is the linear correlation coefficient between variables x and y .

This system can be used to relate the variances in spaces G_P and G_Φ . To do this I first form a linearized relationship between the two spaces. I then have

$$\overline{N}(P/N - \bar{p}) \approx (P - \overline{P}) - \bar{p}(N - \overline{N}) \quad (20)$$

with corresponding variance vectors related by

$$\vec{\sigma}_\Phi \approx \vec{\sigma}_P - \bar{p}\vec{\sigma}_N \quad (21)$$

(or more generally $\bar{m}_2 \vec{\sigma}_{<m>} = \vec{\sigma}_{m_1} - \bar{m}_1/\bar{m}_2 \cdot \vec{\sigma}_{m_2}$). This vector relation is illustrated on the left side of Fig. 8. Examples of global intensive quantities for a hadronic system are N_K/N_π (flavor fluctuations), N_+/N_- (isovector fluctuations), $N_0/(N_+ + N_-)$ (isotensor fluctuations) and $N_{B-\bar{B}}/N_\pi$ (baryon-number fluctuations).

With some manipulation of Eq. (21) we obtain two useful relationships

$$\sigma_P^2 = \sigma_\Phi^2 + 2r_1 \bar{p}\sigma_N \sigma_\Phi + \bar{p}^2 \sigma_N^2 \quad (22)$$

$$\sigma_\Phi^2 = \sigma_P^2 - 2r_2 \bar{p}\sigma_N \sigma_P + \bar{p}^2 \sigma_N^2 \quad (23)$$

where $r_1 = r(<p>, N^2)$ and $r_2 = r(P, N)$ are the linear correlation coefficients. I define angles θ_1 and θ_2 by $r_1 \equiv \cos\theta_1$ and $r_2 \equiv \cos\theta_2$. The geometrical interpretation of these angles is also shown in Fig. 8. By projecting out cartesian components I obtain useful sine and cosine equations

$$\bar{p}\sigma_N = \sigma_P \cos\theta_2 - \sigma_\Phi \cos\theta_1 \quad (24)$$

$$\sigma_\Phi \sin\theta_1 = \sigma_P \sin\theta_2 = \sqrt{N} \sigma_p + \delta\sigma \quad (25)$$

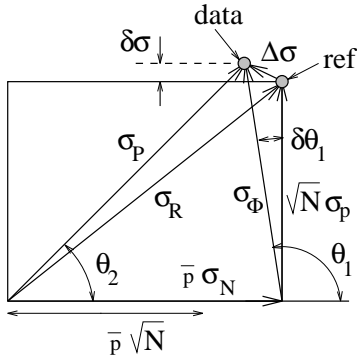
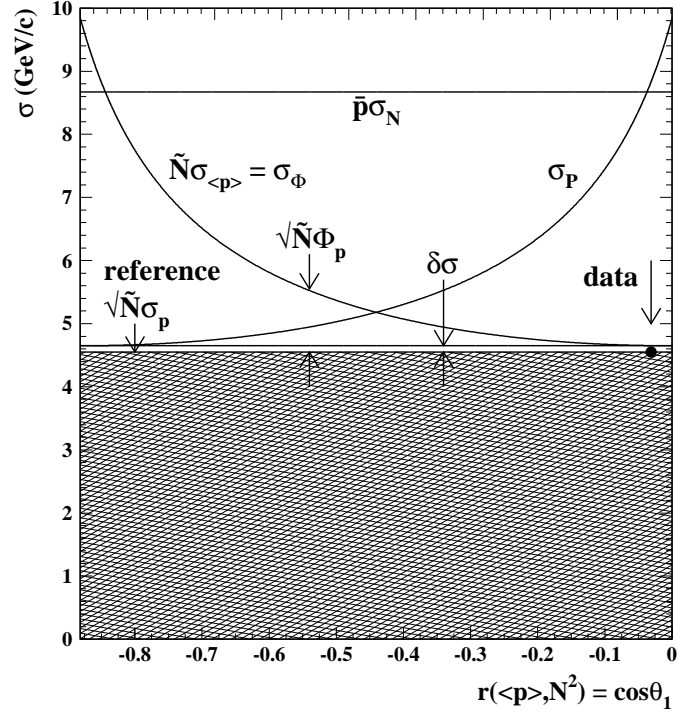


Figure 8: A vector representation relating variances in G_P and G_Φ is shown above. σ_R represents a reference, and σ_P , σ_Φ equivalently represent the data. The dependence of the two variance quantities on a linear correlation coefficient is shown at right.



the second of which includes a possible contribution $\delta\sigma$ from an unspecified variance source. Eqs. (23) and (24) are analogs to Eqs. (17) and (23) respectively of [16]. In the present case the results assume no special conditions and arbitrary deviation from the CLT whereas the results in [16] are derived in the specific context of an equilibrated pion gas.

9.2 Vector deviation from a CLT reference

The vector description of variance in spaces G_P and G_Φ has so far been generic. That is, $-\cos\theta_1$ and $\cos\theta_2$ could have taken any consistent pair of values on the interval $[0, 1]$. The two spaces are complementary and interchangeable. A reference system for the CLT applicable to HI collisions can now be defined by $\vec{\sigma}_R = [\bar{p}\sigma_N, \sqrt{N}\sigma_p]$. This choice implies zero covariance between variables $\langle p_t \rangle$ and N ($\cos\theta_1 = 0$) and momentum variance given by a CLT expectation. The number variance σ_N^2 exceeds the Poisson expectation \bar{N} primarily due to correlations from resonance decays. This reference is consistent with the central limit theorem but represents a more complete description (including covariance). The physical interpretation of this reference is consistent with the CLT – individual particle momenta chosen independently from a fixed parent distribution represented by an inclusive p_t distribution.

Given this reference choice I can define a vector measure of deviation from the CLT reference, vector $\Delta\vec{\sigma}$ running from the reference ($\vec{\sigma}_R$) to a point representing the data ($\vec{\sigma}_P$)

$$\begin{aligned}
 \Delta\vec{\sigma} &\equiv \vec{\sigma}_P - \vec{\sigma}_R \\
 &= [\sigma_\Phi \cos\theta_1, \sigma_\Phi \sin\theta_1 - \sqrt{N}\sigma_p] \\
 &\approx \sqrt{N}[\sigma_p \delta\theta_1, \Phi_{p_t} - \sigma_p \delta\theta_1^2/2] \\
 &\approx \sqrt{N}[\sigma_p \delta\theta_1, \delta\sigma]
 \end{aligned} \tag{26}$$

where we finally retain only terms linear in small quantities $(\delta\theta_1, \delta\sigma)$. We see that *two* quantities, $\delta\theta_1 \approx \cos\theta_1 = lcc \propto B$ and $\delta\sigma \propto \Phi_{p_t}$, which both have the properties 1) – linearity and 2) – zero for independent objects, provide a measure of deviation from a dynamical variance reference relevant to the CLT. This extends the initial concept of the Φ_{p_t} measure finally to handle also N - $\langle p_t \rangle$ correlations.

Dependence of the variances on $r_1 = \cos\theta_1$ is shown on the right side of Fig. 8. Note that Φ_{p_t} depends *quadratically* on $r(\langle p \rangle, N^2) = \cos\theta_1$, which is just the measure sensitive to N - $\langle p \rangle$ correlations. This dependence was first suggested by numerical simulations [27] and implies that Φ_{p_t} itself is ironically quite insensitive to N - $\langle p \rangle$ correlations. Combining Eqs. (17), (21) and (25) I have

$$\begin{aligned}\Phi_{p_t} &\equiv \sigma_\Phi / \sqrt{N} - \sigma_p \\ &\approx \frac{\sigma_p}{2} \cos^2\theta_1 + \delta\sigma.\end{aligned}\tag{27}$$

which is consistent with Eq. (26). We conclude that for nearly symmetric systems ($\vec{\sigma}_P \approx \vec{\sigma}_R$) Φ_{p_t} is primarily sensitive to *variance* sources ($\delta\sigma$), whereas *covariance* sources such as N - $\langle p \rangle$ correlations must be measured explicitly by a covariance or linear correlation coefficient. In Sec. 11 we will consider the relationship between covariance matrices in G_P and G_Φ and similar spaces more generally.

A concern is raised in [24] that $\sigma_{\langle p_t \rangle}^2 \equiv \sigma_{p_t/N}^2 \neq \sigma_\Phi^2 / N^2$, that is Φ_{p_t} is not directly related to $\sigma_{\langle p_t \rangle}^2$. But it is easy to show that the latter quantities in the present case are arbitrarily close, to the extent that the relationship between spaces G_P and G_Φ can be linearized. This depends in turn on the degree to which the CLT conditions are met and on the size of the total multiplicity \overline{N} (equivalently the scale interval from particle to event).

9.3 Significance of A , B and Φ_{p_t}

We now reexamine A , B and Φ_{p_t} in light of this more complete vector variance representation. The relationships among A , B and Φ_{p_t} were first suggested by numerical simulations [27] which demonstrated a strong correlation between A and B when N - $\langle p_t \rangle$ correlations were artificially introduced into a mixed-event reference. A and $-B$ varied together over a $40 \text{ MeV}/c$ range while Φ_{p_t} fluctuated about zero. This suggested either that A and B were redundant or that one carried a subset of the information contained in the other. The relationship $A + B \approx 2\overline{N}\sigma_p\Phi$ simply defines a plane in a 3-D space of (A, B, Φ) without giving emphasis to any one of the three related quantities. We would like to develop a more substantial understanding of this relationship.

Given the definitions of A , B and Φ I can express A as

$$\begin{aligned}A &\approx \sigma_\Phi^2 - \overline{N}\sigma_p^2 + 2\overline{p}\sigma_N\sqrt{N}\sigma_p\cos\theta_1 \\ &\approx 2\sqrt{N}\sigma_p \cdot \sqrt{N}\Phi + 2\overline{p}\sigma_N \cdot \sqrt{N}\sigma_p\cos\theta_1 \\ &\approx 2\Delta\vec{\sigma} \cdot \vec{\sigma}_R\end{aligned}\tag{28}$$

A thus has the form of a vector dot product. Note that if the deviation $\Delta\vec{\sigma}$ is orthogonal to the reference $\vec{\sigma}_R$ there is no *net* two-point correlation, even though the CLT is not satisfied. We now have a relationship between departures from the CLT on the one hand ($\cos\theta_1$ and Φ_{p_t}), which are parts of a global measure system, and net two-point correlations on the other (A), which refer to the microscopic particle complement. This vector result *suggests* that A is in a measure class distinct from $\cos\theta_1$ and Φ_{p_t} . We will see below that this is indeed the case. In pursuing the structure of Φ_{p_t} we have discovered a broader organization.

We also find that the presence of two-point correlations *may* result in deviation from a reference variance system, but not necessarily. On the other hand deviation from a reference system *may* result in net two-point correlation, but again there is no necessary relationship. Variance changes and $N-\langle p_t \rangle$ correlations (covariances) are global effects with possible thermodynamical interpretations. Both *may* correspond to net two-point correlations at the microscopic level or *vice versa*, but there is no necessary connection. This picture is still not fundamental. We require a more complete treatment of variance and correlation, including explicit scale dependence, to fully interpret any results.

This treatment of variance provides a geometrical model relating archetypal spaces G_P and G_Φ . It illustrates how small incremental contributions to variance and covariance relative to a reference are related to the Φ_{p_t} measure. It illustrates dramatically the reason that Φ_{p_t} is insensitive to $N-\langle p_t \rangle$ correlations, which are instead measured by a covariance element. We have established that the decomposition of Φ_{p_t} into components $A+B$ is nontrivial, that in fact Φ_{p_t} itself approximates a component of a more comprehensive measure system, and that its value is *not* related in a necessary way to the presence or absence of system correlations, either differential or integral. To proceed further we must extend our description to a system of scale-dependent covariance matrices.

10 Scale-Dependent Variance

To address the ambiguities encountered in the Φ_{p_t} analysis we now attempt to extract more complete information from collision events. We do this by expanding the description in two ways: 1) we move to an explicit scale-dependent description, and 2) we adopt a description based on the complete covariance matrix for all additive measures relevant to the dynamical system. The vector representation of variance developed previously is contained in a larger description involving scale-dependent covariance matrices and correlation integrals.

There are four spaces involved in a variance study. $P_1(\vec{x})$ is the single-point primary space with measure-density system \vec{m} defined on it. $Q_2(\vec{x} \otimes \vec{x})$ is the corresponding two-point space with measure product density $\vec{m} \otimes \vec{m}$ defined on it. These spaces may not be directly accessible to observation. Measure *integrals* for specific partition systems are themselves distributed on spaces $G_1(\vec{m})$ and $G_2(\vec{m} \otimes \vec{m})$ which may represent the results of direct measurement. The covariance matrices described below represent a gaussian approximation to distributions on $G_1(\vec{m})$. Covariance-matrix differences are in turn related to correlation integrals $\Delta C_2(m_1, m_2; \delta x_1, \delta x_2)$ over two-point space $Q_2(\vec{x} \otimes \vec{x})$ and covariance integrals $A(\delta x_1, \delta x_2; m_1, m_2)$ over space $G_2(\vec{m} \otimes \vec{m})$.

10.1 Variance and correlation integrals

A variance is implicitly defined at a particular scale (the bin size of a binning system). Each entry in Fig. 7 corresponds to an event, which can also be considered as a partition element or ‘bin’ in some larger space (Sec. 3) resulting from the sampling of a general thermodynamic system at some arbitrary scale. We now move to a more explicit and general scale-dependent binning approach, with events and particles considered as particular cases.

I assume a space Ω in which particles, events and event ensembles are arrayed. Ω is the primary or P space. This space is binned at some arbitrary scale δx . At some smaller scale a bins may be identified with particles. At a larger scale L bins may coincide with events. The space is arbitrarily large and may contain many events (an event ensemble). Associated with any P space is a set of Q_q spaces consisting of q -fold cartesian products of P space. In the discussion of variances that

follows the space Q_2 will be of primary importance. This space represents all the point pairs of P space.

I consider relationships among measures on a bounded scale interval $[\delta x_1, \delta x_2]$ on which I define a scale-dependent covariance matrix $\mathbf{K}(\vec{m}, \delta x)$. (\vec{m} is the set of measures defined on Ω). This scale-local generalization with particles and events as limiting cases provides a more complete and self-consistent description of variance. I begin with the scale dependence of variance for a single measure.

Consider an additive measure (say momentum) distributed on Ω and binned at two scales $(\delta x_1, \delta x_2)$, with typical bin contents $p(\delta x_1)$ and $p(\delta x_2)$ and with $p(\delta x_2) = \sum_{i=1}^N p_i(\delta x_1)$ for the smaller bins contained in a larger bin. I can then write for each large-scale bin

$$p^2(\delta x_2) = \sum_{i=1}^N p_i^2(\delta x_1) + \sum_{i \neq j=1}^N p_i(\delta x_1) p_j(\delta x_1) \quad (29)$$

If I average over all bins I have

$$\overline{p^2}(\delta x_2) - N \overline{p^2}(\delta x_1) = \overline{\sum_{i \neq j} p_i(\delta x_1) p_j(\delta x_1)} \quad (30)$$

where $N(\delta x_1, \delta x_2) = (\delta x_2 / \delta x_1)^d = M(\delta x_1) / M(\delta x_2)$ is the number of smaller bins contained in a larger bin (d is the P-space dimension, which I hereafter omit to simplify notation). The RHS is also expressible as the integral over pair-momentum space of a two-point density distribution. I discuss this integral in more detail below. The LHS can be expressed as the difference between two rank-2 correlation integrals.

The rank- q normalized correlation integral at scale δx for an additive measure p can be approximated by

$$C_q(\delta x) \approx \sum_{i=1}^{M(\delta x)} p_i^q(\delta x) / \left\{ \sum_{i=1}^{M(\delta x)} p_i(\delta x) \right\}^q \quad (31)$$

where $M(\delta x)$ is the number of *occupied* bins at scale δx in the space Ω (containing nonzero measure). This bin-based definition approximates an integral on scale in space Q_q from zero scale up to the bin size δx . For an uncorrelated or uniform distribution $C_q(\delta x) \rightarrow C_{q,ref} = M^{1-q}(\delta x)$, which is one possible reference choice.

I can express mean-square quantities in terms of the correlation integral by

$$\overline{p^2}(\delta x) \approx M(\delta x) \overline{p^2}(\delta x) C_2(\delta x) \quad (32)$$

from which I obtain

$$\begin{aligned} \overline{p^2}(\delta x_2) - N \overline{p^2}(\delta x_1) &\approx M(\delta x_2) \overline{p^2}(\delta x_2) C_2(\delta x_2) - [M^2(\delta x_1) / M(\delta x_2)] \overline{p^2}(\delta x_1) C_2(\delta x_1) \\ &\approx \overline{p^2}(\delta x_2) M(\delta x_2) \{C_2(\delta x_2) - C_2(\delta x_1)\} \\ &\approx \overline{p^2}(\delta x_2) M(\delta x_2) C_2(\delta x_1, \delta x_2) \end{aligned} \quad (33)$$

The difference between correlation integrals at two scales is just a correlation integral over a scale interval, $C_2(\delta x_1, \delta x_2)$. This justifies the observation made earlier that at least some global variables, those based on moments, are scale integrals. If we subtract from both sides of Eq. (33) the

corresponding expressions for a reference distribution such as a uniform distribution, with $\overline{p^2} = \bar{p}^2$ and $\overline{\sum_{i \neq j} p_i p_j} = N(N-1)\bar{p}^2$ as shown in Fig. 9, then we have the variance difference

$$\sigma^2(p, p, \delta x_2) - N\sigma^2(p, p, \delta x_1) = \bar{p}^2(\delta x_2)M(\delta x_2)\Delta C_2(p, p, \delta x_1, \delta x_2) \quad (34)$$

where we adopt the notation $(m_1, m_2, \delta x)$ to anticipate generalization to covariance matrices for arbitrary measure systems. $\Delta C_2(p, p; \delta x_1, \delta x_2)$ is then the difference between correlation integrals over a bounded scale interval for object and reference distributions. The implicit reference for a standard variance is a uniform distribution on the system of occupied bins or *support* of the object distribution. The variance quantities $\sigma^2(p, p, \delta x)$ could also be defined in terms of an arbitrary nonuniform reference.

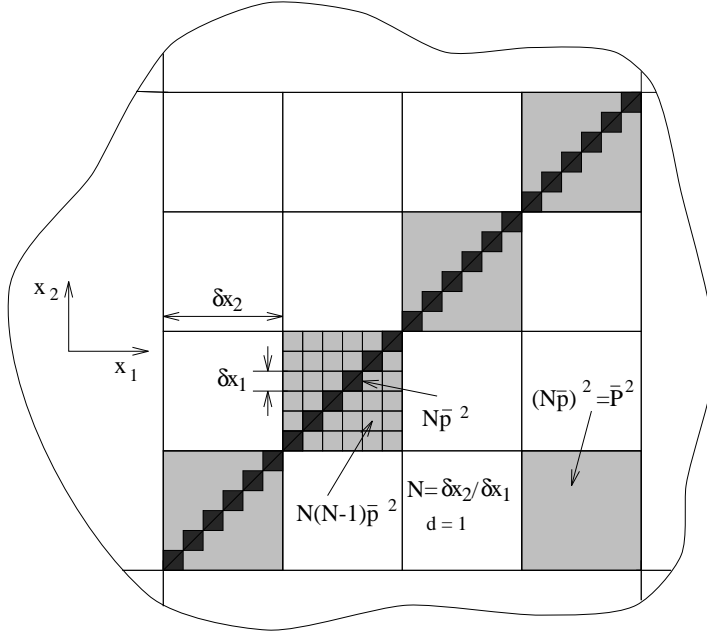


Figure 9: Two-point space $Q_2(\vec{x} \otimes \vec{x})$ corresponding to a 1-D primary space. Statistical quantities are given for a uniform reference distribution. Q_2 is binned at three arbitrary scales. A comparison is made in the text between the variance contained in bins at a smaller scale (dark) and at a larger scale (light).

Fig. 9 illustrates several of these concepts. Bins at smaller (dark) and larger (light) scale are highlighted along the main diagonal of $Q_2(\vec{x} \otimes \vec{x})$ space. The variation of *total variance* (next section) with scale implicit in Eq. (34) can be illustrated with the following argument. $\sigma^2(p, p, \delta x_1)$ represents the variance per bin for the smaller bins, while $\sigma^2(p, p, \delta x_2)$ represents the same quantity for the larger bins. Since there are N smaller bins in a larger bin $\sigma^2(p, p, \delta x_2) - N\sigma^2(p, p, \delta x_1)$ represents the *excess* variance in a larger bin not accounted for by that contained in the smaller bins. This excess must be contributed by the $N(N-1)$ off-diagonal smaller bins contained in a larger bin. Within a factor, the measure of additional variance in the off-diagonal bins is the correlation integral difference $\Delta C_2(p, p; \delta x_1, \delta x_2)$ over the scale interval bounded by the two bin sizes. The two diagonal arrays of bins approximate the integration regions (diagonal strips) appropriate for exact correlation integrals $C_2(\delta x_1)$ and $C_2(\delta x_2)$ in Q_2 . This connection of variance with correlation integrals is in part motivated by subsequent integration of covariance treatments with scale-local topological measures [2, 3, 28].

I now return to the cross term

$$\overline{\sum_{i \neq j} p_i p_j} \approx \int p_1 p_2 \cdot \rho_2(p_1, p_2) dp_1 dp_2 \quad (35)$$

This relation connects the two-point measure space $G_2(p, p)$ (right side) with the two-point space $Q_2(x, x)$ represented in Fig. 9 (left side). The integral over the entire two-point momentum space

of the density $\rho_2(\delta x_1, \delta x_2; p_1, p_2)$ is equal to a correlation integral over a *specific scale interval* in Q_2 of the density $\rho_2(p, p; x_1, x_2)$. Distribution $\rho_2(\delta x_1, \delta x_2; p_1, p_2)$ is a parametric function of that scale interval as indicated by the argument. Note the convention $\rho_2(\text{params}; \text{variables})$ where *params* determine the nature of the density and *variables* are the position indices for the corresponding space.

It is useful to adopt the convention that symbol a represents a ‘particle’ scale and L represents an ‘event’ scale, with Ω representing the entire containing space. In the particle-event limit $N(\delta x_1, \delta x_2) \rightarrow \bar{N}_{part}$, and $M(\delta x_2) \rightarrow \mathcal{M}$ (total event number) as $(\delta x_1, \delta x_2) \rightarrow (a, L)$. The distribution in $G_P = G_1(p, n)$ is then $\rho_2(L, \Omega; p, n)$, and the two-point particle momentum distribution in $G_2(p_1, p_2)$ is $\rho_2(a, L; p_1, p_2)$. We can generally write $\rho_2(\delta x_1, \delta x_2; m_i, m_j)$ for the density in $G_2(m_1, m_2)$, consistent with the notation above. To complete the treatment we write

$$\begin{aligned} \sigma^2(p, p, \delta x_2) - N\sigma^2(p, p, \delta x_1) &= \left\{ \overline{\sum_{i \neq j} p_i p_j} \right\}_{obj} - \left\{ \overline{\sum_{i \neq j} p_i p_j} \right\}_{ref} \\ &\approx \int p_1 p_2 \cdot \{ \rho_{2,obj}(\delta x_1, \delta x_2;) - \rho_{2,ref}(\delta x_1, \delta x_2;) \} dp_1 dp_2 \\ &\equiv A(\delta x_1, \delta x_2; p, p) \end{aligned} \quad (36)$$

which defines $A(\delta x_1, \delta x_2; p, p)$ as an integral over the two-point measure space $G_2(p_1, p_2)$, the same A which appears in the decomposition of Φ_{pt} . In case $\rho_{2,ref}(\delta x_1, \delta x_2;)$ corresponds to a uniform reference then $A(\delta x_1, \delta x_2; p, p)$ is the *covariance* of the two-point density $\rho_{2,obj}(\delta x_1, \delta x_2; p_1, p_2)$.

This result for variance $\sigma^2(p, p, \delta x)$ applies also to other elements of any covariance matrix for a measure system \vec{m} . Thus, the covariance difference $\sigma^2(p, n, \delta x_2) - N\sigma^2(p, n, \delta x_1)$ equals an integral over the space $G_2(p, n)$ of a corresponding two-point density difference $\rho_{2,obj}(\delta x_1, \delta x_2; p, n) - \rho_{2,ref}(\delta x_1, \delta x_2; p, n)$ represented by $A(\delta x_1, \delta x_2; p, n)$.

10.2 Total variance

The linearity property invoked in the derivation of Φ_{pt} is actually an implicit invocation of *scale invariance*. The approximately invariant measure is the total variance introduced in the previous section. Recalling that $N(\delta x_1, \delta x_2) = M(\delta x_1)/M(\delta x_2)$ and multiplying through Eq. (34) by $M(\delta x_2)$ (omiting subscripts) I have

$$\Sigma^2(p, p, \delta x_2) - \Sigma^2(p, p, \delta x_1) \approx P^2 \Delta C_2(p, p; \delta x_1, \delta x_2) \quad (37)$$

where $\Sigma^2(p, p, \delta x) \equiv M(\delta x) \sigma^2(p, p, \delta x)$ is the *total variance* of a measure p on Ω relative to a uniform reference at scale δx ($\sigma^2(p, p, \delta x)$ is the *per-bin* variance), and $P \equiv M(\delta x) \bar{p}(\delta x)$ is a scale-invariant *total measure* on Ω . We establish here the concept that variation of total (co)variance on scale is the underlying issue for the central limit theorem and any correlation analysis based on global variables. Variation of the scale-dependence of total variance may be due to changes in the underlying system Lagrangian for certain constraint configurations (the neighborhood of a phase boundary), in which a correlation onset may appear or disappear, producing large fluctuations in the primary *DoF* which determine the basic statistical fluctuations for all observables.

We can connect increased total variance with its source as represented by the correlation integral by developing a differential equation. We simply take Eq. (37) to the limit of small scale interval

$$\partial \Sigma^2(p, p, \delta x) / \partial \delta x \approx P^2 \Delta \partial C_2(p, p; \delta x) / \partial \delta x \quad (38)$$

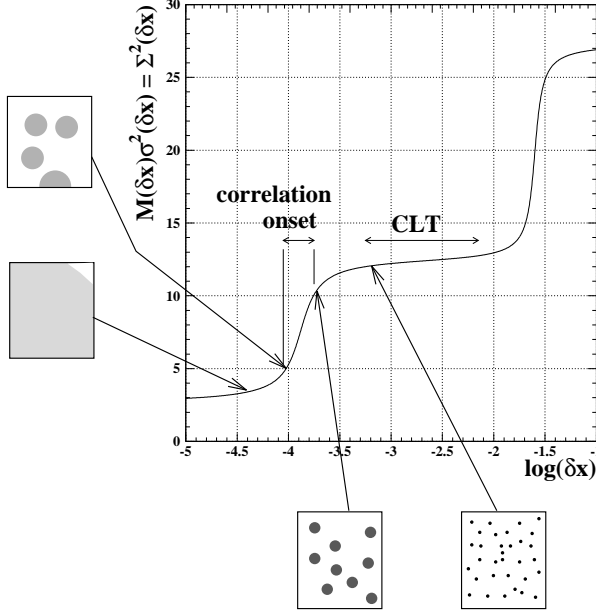


Figure 10: Scale-dependence of total variance for a system with scale-localized correlation onsets. Correlation onset extends from the characteristic size of an object to the mean spacing between objects where localized objects predominate. The distribution of total variance is not necessarily monotone increasing with scale due to the possibility of anticorrelation in distributions. Such curves may vary with global constraints such as (T, μ) near a phase boundary, leading to departure from a CLT expectation.

where $\partial C_2(p, p; \delta x)/\partial \delta x$ is an autocorrelation density. This relation then implies that any variation in total variance over a scale interval (the object of a CLT test) corresponds to a nonzero difference between object and reference autocorrelation densities over that interval. Since the autocorrelation density is a quantitative symmetry measure we see that scale-invariance of total variance is intimately related to the scale-dependent *relative symmetry* of object and reference distributions.

The CLT provides a useful reference for measure systems in which localized regions of correlation onset are separated by significant scale intervals with negligible accumulation of total variance (approximate scale invariance) as illustrated in Fig. 10. Distinguishable objects that result from a localized correlation onset can be designated ‘particles’ or dynamical *DoF*, and correlation analysis based on integral quantities (global variables) is then simplified to characterizing any (relatively small) total variance accumulated over some scale interval between these correlation onsets.

Approximate invariance of total variance over a bounded scale interval is expressed by $\Sigma^2(\delta x_2) \approx \Sigma^2(\delta x_1)$ for some $[\delta x_1, \delta x_2]$. We then have

$$\begin{aligned}
 N(\delta x_1, \delta x_2) \sigma_{\langle p_t \rangle}^2 &\equiv \sigma_{P_t}^2 / N(\delta x_1, \delta x_2) \\
 &= \Sigma^2(\delta x_2) / M(\delta x_1) \\
 &\approx \Sigma^2(\delta x_1) / M(\delta x_1) \\
 &\approx \sigma_{p_t}^2
 \end{aligned} \tag{39}$$

which is a statement of the CLT as used in the definition of the Φ_{p_t} variable (Eq. (1)).

In summary, the *accumulation* of total variance on scale is a basic issue for any global-variables analysis based on a CLT test. This is simplest to study in cases where one has scale-localized correlation onsets separated by scale intervals of nearly constant total variance, over which a CLT reference is a good approximation. We have shown - Eq. (36) - that the difference $\sigma_P - N \sigma_p$ is actually a scale integral represented by $A(\delta x_1, \delta x_2; p, p)$, and that it is desirable to examine the integrand of A directly in a differential analysis whenever possible. An extension of this program to *scale-local* topological measures examines the detailed scale-dependence of total-variance accumulation

or its equivalent without prior assumptions [5]. With a scale-dependent treatment of variance in hand we now proceed to a general system of scale-dependent covariance matrices.

11 Scale-Dependent Covariance Matrices

We now consider an arbitrary system of dynamical measures and the corresponding scale-dependent covariance matrix. A covariance matrix provides a gaussian approximation to a scale-dependent density distribution in measure space $G_1(\vec{m})$. As we have shown, variation of (co)variances on scale can be equated to corresponding integrals over two-point measure spaces. We distinguish between correlation integrals of densities $\rho_2(m_1, m_2; x_1, x_2)$ on spaces $Q_2(x_1, x_2)$ represented by $C_2(m_1, m_2; \delta x_1, \delta x_2)$ and integrals of densities $\rho_2(\delta x_1, \delta x_2; m_1, m_2)$ on two-point measure spaces $G_2(m_1, m_2)$ represented by $A(\delta x_1, \delta x_2; m_1, m_2)$.

11.1 Single-point and two-point measure spaces

If a space $P_1(\vec{x})$ supports a set of additive measures $\vec{m} = (m_1, m_2, \dots)$ (e.g., momentum p and multiplicity n) having bin values $\vec{m}_i(\delta x)$ for the i th bin of size δx represented as a density in a single-point measure space $G_1(\vec{m})$ with projections onto pairs of measure values such as $G_1(p, n)$, and I form a two-point space $Q_2(\vec{x} \otimes \vec{x})$ (Fig. 9) then I can define a corresponding general two-point measure space $G_2(\vec{m}_1 \otimes \vec{m}_2)$ with projections onto pair spaces such as $G_2(p_1, p_2)$, $G_2(p_1, n_2)$, $G_2(n_1, n_2)$ and so forth. Each bin in Q_2 or pair of bins in P space contributes one ‘point’ (integral of the measure densities over the bin) to a density distribution $\rho_2(\delta x_1, \delta x_2; m_1, m_2)$ for a specific scale interval in each of the two-point measure spaces (these spaces are binned also, at the resolution of a detection device). These density distributions are parameterized by the scale intervals in P space used to define them.

Specifying a scale interval $(\delta x_1, \delta x_2)$ may seem mysterious. The lower limit is obviously the bin size in P space used to define the densities. But to form a correlation space one must specify over what region of P space bin contents will be correlated (Fig. 9). The size of this region (a type of correlation length) provides the upper limit to the scale interval. In an EbyE analysis ‘particles’ are correlated within ‘events’ and not between events over an entire ensemble.

Measures defined at the event scale may be used to sort events into categories. One aspect of EbyE analysis is to establish such categories. Subsequent EbyE analysis may indicate that a measure space is incomplete by revealing previously unknown correlation structures within events. In defining references for correlation analysis it is desirable to form reference pairs from events that are ‘nearest neighbors’ in this expanded measure space, for instance, events with the closest total multiplicities or transverse energies.

The quantity $A(\delta x_1, \delta x_2; p, p)$ defined in Eq. (36) is an example of an integral over a two-point measure space. It is just the term A identified in the decomposition of the Φ_{pt} measure. It is the integral over measure space $G_2(p_1, p_2)$ of the connected two-point correlation function $\rho_{2,obj}(a, L; p_1, p_2) - \rho_{2,ref}(a, L; p_1, p_2)$ (Fig. 5). Similar integrals for various additive measures taken pair-wise form the elements of the integral matrix $\mathbf{A}(\delta x_1, \delta x_2; \vec{m} \otimes \vec{m})$. The relationship between distribution moments and q -point densities is also discussed in [29].

11.2 Covariance-matrix elements

Given a general measure system \vec{m} defined on Ω and arguing by analogy with the momentum-variance results above, I can write for any pair of component measures m_1 and m_2 the quadratic

mean

$$M(\delta x) \overline{m_1(\delta x) m_2(\delta x)} = M^2 \bar{m}_1 \bar{m}_2 C_2(m_1, m_2; \delta x) \quad (40)$$

where $C_2(m_1, m_2; \delta x)$ is the correlation integral up to scale δx of the two-point density distribution $\rho_2(m_1, m_2; x_1, x_2)$ on Q_2 . Note that on the left side (m_1, m_2) are measure values from the same bin in P_1 whereas on the right they come from different bins (off-diagonal element of Q_2). The corresponding total (co)variance is given by

$$\begin{aligned} M(\delta x) \sigma^2(m_1, m_2, \delta x) &= M(\delta x) \{ \overline{m_1(\delta x) m_2(\delta x)} - \overline{m_1(\delta x)} \cdot \overline{m_2(\delta x)} \} \\ &\equiv \Sigma^2(m_1, m_2, \delta x) \\ &= M^2 \bar{m}_1 \bar{m}_2 \Delta C_2(m_1, m_2; \delta x) \end{aligned} \quad (41)$$

A comparison of quadratic means at different scales is given by

$$M(\delta x_2) \overline{m_1(\delta x_2) m_2(\delta x_2)} - M(\delta x_1) \overline{m_1(\delta x_1) m_2(\delta x_1)} = M^2 \bar{m}_1 \bar{m}_2 C_2(m_1, m_2; \delta x_1, \delta x_2) \quad (42)$$

and a (co)variance comparison (generalized form of CLT) is given by

$$\begin{aligned} \sigma^2(m_1, m_2, \delta x_2) - N(\delta x_1, \delta x_2) \sigma^2(m_1, m_2, \delta x_1) &= \bar{m}_1 \cdot \bar{m}_2 M(\delta x_2) \Delta C_2(m_1, m_2; \delta x_1, \delta x_2) \\ &= \int m_1 m_2 \{ \rho_{2,obj}(\delta x_1, \delta x_2; m_1, m_2) - \rho_{2,ref}(\delta x_1, \delta x_2; m_1, m_2) \} dm_1 dm_2 \\ &\equiv A(\delta x_1, \delta x_2; m_1, m_2), \end{aligned} \quad (43)$$

or in terms of total (co)variance by

$$\begin{aligned} \Sigma^2(m_1, m_2, \delta x_2) - \Sigma^2(m_1, m_2, \delta x_1) &= M^2 \bar{m}_1 \bar{m}_2 \Delta C_2(m_1, m_2; \delta x_1, \delta x_2) \\ &\equiv \mathcal{A}(\delta x_1, \delta x_2; m_1, m_2). \end{aligned} \quad (44)$$

The quantities $\Sigma^2(m_1, m_2, \delta x)$ are elements of a total covariance matrix $\mathcal{K}(\vec{m}, \delta x)$, The per-bin variances and covariances $\sigma^2(m_1, m_2, \delta x)$ are elements of the usual covariance matrix $\mathbf{K}(\vec{m}, \delta x)$. The quantities $\bar{m}_1 \cdot \bar{m}_2 M(\delta x_2) \Delta C_2(m_1, m_2; \delta x_1, \delta x_2)$ are equal to corresponding elements of the integral matrix $\mathbf{A}(\delta x_1, \delta x_2; \vec{m} \otimes \vec{m})$, and the quantities $M^2 \bar{m}_1 \bar{m}_2 \Delta C_2(m_1, m_2; \delta x_1, \delta x_2)$ are equal to elements of an integral matrix $\mathcal{A}(\delta x_1, \delta x_2; \vec{m} \otimes \vec{m})$, both corresponding to a specific scale interval. We can summarize with the matrix equations

$$\mathbf{K}(\vec{m}, \delta x_2) - N(\delta x_1, \delta x_2) \mathbf{K}(\vec{m}, \delta x_1) = \mathbf{A}(\delta x_1, \delta x_2; \vec{m} \otimes \vec{m}) \quad (45)$$

and

$$\mathcal{K}(\vec{m}, \delta x_2) - \mathcal{K}(\vec{m}, \delta x_1) = \mathcal{A}(\delta x_1, \delta x_2; \vec{m} \otimes \vec{m}) \quad (46)$$

which relate covariance matrix elements and integrals over two-point measure spaces for a bounded scale interval.

11.3 Covariance matrices in G_P and G_Φ

We now apply these new concepts to the original problem of understanding the relationship among A , B and Φ_{pt} . The spaces $G_P \equiv G_1(p, \bar{p}n)$ and $G_\Phi \equiv G_1(\bar{n}p/n, \bar{p}n) \approx G_1(p - \bar{p}n, \bar{p}n)$ shown in Fig. 7 provide an archetypal system for the study of extensive/intensive variable combinations

and associated covariance structure. These spaces are approximately related by a linear shear transformation \mathbf{S} . The corresponding covariance matrices \mathbf{K}_P and \mathbf{K}_Φ for an arbitrary distribution common to the two spaces are then defined by the relations

$$\begin{aligned}
 \mathbf{K}_\Phi &= \begin{pmatrix} \sigma_\Phi^2 & \bar{p}\sigma_{\langle p \rangle N^2}^2 \\ \bar{p}\sigma_{\langle p \rangle N^2}^2 & \bar{p}^2\sigma_N^2 \end{pmatrix} \\
 &\approx \tilde{\mathbf{S}} \cdot \mathbf{K}_P \cdot \mathbf{S} \\
 &\approx \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sigma_P^2 & \bar{p}\sigma_{PN}^2 \\ \bar{p}\sigma_{PN}^2 & \bar{p}^2\sigma_N^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
 &\approx \begin{pmatrix} \sigma_P^2 - 2\bar{p}\sigma_{PN}^2 + \bar{p}^2\sigma_N^2 & \bar{p}\sigma_{PN}^2 - \bar{p}^2\sigma_N^2 \\ \bar{p}\sigma_{PN}^2 - \bar{p}^2\sigma_N^2 & \bar{p}^2\sigma_N^2 \end{pmatrix} \quad (47)
 \end{aligned}$$

These matrix equations compactly represent the relationships among covariance matrix elements derived in Sec. 9.1 using a vector representation.

I now define a reference covariance matrix at small (particle) scale in G_Φ which is analogous to $\vec{\sigma}_R$ in the vector variance formulation and which may serve as a null hypothesis based on a physical model. I obtain the corresponding form in G_P by applying the same shear transformation as before. The reference matrix in G_Φ has zero covariance and variances σ_p^2 and σ_n^2 determined by some unspecified elementary phenomena. For $\delta x_1 \approx a$, σ_p^2 may correspond to a thermal distribution and $\sigma_n^2 \approx 1$ modulo quantum statistics. The momentum \hat{p} represents a characteristic momentum at small scale which may coincide with \bar{p} .

$$\mathbf{K}_\Phi(\delta x_1) = \begin{pmatrix} \sigma_p^2 & 0 \\ 0 & \hat{p}^2\sigma_n^2 \end{pmatrix} \quad (48)$$

This choice is particularly simple and could represent an independent-particle-emission hypothesis. We can now obtain expressions for covariance difference matrices in both spaces.

$$\mathbf{K}_\Phi(\delta x_2) - N\mathbf{K}_\Phi(\delta x_1) = \begin{pmatrix} \sigma_\Phi^2 - N\sigma_p^2 & \hat{p}\sigma_{\langle p \rangle N^2}^2 \\ \hat{p}\sigma_{\langle p \rangle N^2}^2 & \hat{p}^2(\sigma_N^2 - N\sigma_n^2) \end{pmatrix} \quad (49)$$

Using the inverse transform $\mathbf{K}_P(\delta x) = \tilde{\mathbf{S}}^{-1} \cdot \mathbf{K}_\Phi(\delta x) \cdot \mathbf{S}^{-1}$ I can write

$$\begin{aligned}
 \mathbf{K}_P(\delta x_2) - N\mathbf{K}_P(\delta x_1) &= \begin{pmatrix} \sigma_P^2 - N\sigma_p^2 - \hat{p}^2 N\sigma_n^2 & \hat{p}\sigma_{PN}^2 - \hat{p}^2 N\sigma_n^2 \\ \hat{p}\sigma_{PN}^2 - \hat{p}^2 N\sigma_n^2 & \hat{p}^2(\sigma_N^2 - N\sigma_n^2) \end{pmatrix} \\
 &= \begin{pmatrix} \sigma_\Phi^2 - N\sigma_p^2 + 2\hat{p}\sigma_{\langle p \rangle N^2}^2 & \hat{p}\sigma_{\langle p \rangle N^2}^2 + \hat{p}^2(\sigma_N^2 - N\sigma_n^2) \\ \hat{p}\sigma_{\langle p \rangle N^2}^2 + \hat{p}^2(\sigma_N^2 - N\sigma_n^2) & \hat{p}^2(\sigma_N^2 - N\sigma_n^2) \end{pmatrix} \quad (50)
 \end{aligned}$$

I can now relate covariance matrix elements in the space $G_P = G_1(p, n)$ to corresponding two-point measure integrals in the space $G_2((p, n) \otimes (p, n))$ by the matrix equation

$$\begin{aligned}
 \mathbf{K}(p, n, \delta x_2) - N(\delta x_1, \delta x_2) \mathbf{K}(p, n, \delta x_1) &= \mathbf{A}(\delta x_1, \delta x_2; (p, n) \otimes (p, n)) \\
 &= \begin{pmatrix} A_{pp} & \hat{p}A_{pn} \\ \hat{p}A_{pn} & \hat{p}^2 A_{nn} \end{pmatrix} \quad (51)
 \end{aligned}$$

The LHS is the scale difference between covariance matrices for two-point densities defined on subspaces of $G_1(\vec{m})$ (measure values taken from the *same* bin). The RHS represents covariance

integrals for two-point densities defined on subspaces of $G_2(\vec{m} \otimes \vec{m})$ (measure values taken from *different* bins), the most integral correlation measures that can be defined on G_2 . Thus, (co)variances of measures defined on single bins in a primary space are interpreted by two-point correlations among measures on the primary space as determined by covariances on pairs of bins.

We now return to the original problem of A , B and Φ_{pt} . Combining Eqs. (50) and (51) and recalling that $\sigma_\Phi^2 - N\sigma_p^2 \approx 2N\sigma_p\Phi_{pt}$, and $2\hat{p}\sigma_{\langle p \rangle N^2}^2 = -B$ we can now relate the upper left elements of both sides

$$2N\sigma_p\Phi_{pt} - B = A(a, L; p, p) \quad (52)$$

justifying the earlier *ad hoc* decomposition of Φ_{pt} (Sec. 8.2) and clarifying its basis. We see that variance sources (Φ_{pt}) and covariance sources (B) may both correspond to net two-point correlations (A). Φ_{pt} is a ‘part’ of A and not the reverse. This representation only hints at the richness of the system of scale-local correlation measures, even when restricted to pair correlations and (co)variance. Such structures also exist, albeit with increasing algebraic complexity, for arbitrary q -tuples and associated higher moments.

12 Experimental Applications

The previous material is quite general and is intended to accommodate at least conceptually the transition of any dynamical system across a phase boundary near which the dynamical *DoF* are ill-defined. However, experimentally we must contend with the hadronic final state of HI collisions and the measures that can be formed there. We now consider some practical measures and analyses, with examples recently proposed in the literature.

12.1 Available measures and analysis types in the hadronic final state

In the hadronic final state the basic event-wise measures \vec{m} are particle abundances and momenta, $n_{ij}(\delta x)$ and $p_{ij}(\delta x)$, where i refers to particle species and j indicates a bin index. Special cases include $n_{ij}(a) = 1$, $n_{ij}(L) = N_i$ – event abundance for species i , $p_{ij}(a) =$ particle momentum and $p_{ij}(L) = P_i$ – event total momentum for species i . These constitute the primary measures in $\vec{m} = \{m_{ij}\}$. We can also add an integer index n randomly assigned to particles.

A secondary measure system is derived from correlation analysis of individual events. Distribution statistics ($\langle p_t \rangle$) and model parameters (slope parameter) are simple examples. HBT correlation analysis in principle provides event-wise estimates of source radii, flow analysis can estimate the reaction plane and azimuthal anisotropies, and more general event-wise correlation analysis can provide a complete topological assessment of each event in terms of information or dimension transport. A minimal set of correlation parameters which completely characterizes the event can then be included in \vec{m} .

Depending on the abundance of a given particle species one can pursue an integral or differential analysis. In the case of limited abundance one can compare total or per-bin covariance matrix elements at two scales, element by element. These are just central-limit tests as represented by Eq. (51). In the case of greater abundance one can examine directly the integrands of elements of $\mathbf{A}(\delta x_1, \delta x_2; \vec{m} \otimes \vec{m})$. One can analyze differences $\rho_{2,obj}(\delta x_1, \delta x_2; m_1, m_2) - \rho_{2,ref}(\delta x_1, \delta x_2; m_1, m_2)$ or preferably density ratios $\rho_{2,obj}/\rho_{2,ref}$ to search for departures from a reference.

12.2 Partition systems

The partition systems on P space directly accessible in the hadronic final state are at the particle (a) and event (L) scales. Additional partitions can only be defined on $\{\vec{m}, n\}$ and therefore only indirectly qualify as partitions on scale, to the extent that there are correlations between elements of $\{\vec{m}, n\}$ and P space. For instance, strong correlation between longitudinal momentum (rapidity) and axial position means that a partition on rapidity or pseudorapidity provides some ability to partition P space at different scales on axial position. However the scale interval is limited because this correlation is not perfect due to thermal and other fluctuations in longitudinal momentum about the basic Bjorken expansion correlation. Results obtained with such a partition must be carefully interpreted over a restricted scale interval.

A partition on azimuth angle is also possible, but its interpretation depends on the degree of correlation between azimuth angle in momentum space and configuration space. Depending on the emission model this could be very strong (emission from an opaque source) or negligible (emission from a transparent source). Partition of the random index n is the most indirect but also least biased partition method, but its relationship to a scaled partition is not clear in general.

12.3 Nonstatistical fluctuations

The distinction between nonstatistical (‘dynamical’) and statistical fluctuations or variance can be seen as insubstantial given the discussion of scale-dependent variance presented above. ‘Statistical’ variance is accumulated below and within some scale interval characteristic of a correlation onset, or below the scale interval of interest. Dynamical fluctuations as discussed here simply correspond to variance accumulated on a complementary (larger) scale interval – the one of immediate interest. In a different analysis some statistical fluctuations might be considered dynamical by the same argument. For discussion purposes we adopt a distinction of convenience between statistical variance and its complement on scale.

The term A in Eq. (52), which represents the most global aspect of nonstatistical fluctuations, can be decomposed into several contributions as follows

$$\begin{aligned}
 A(a, L; p, p) \approx & \{ \bar{N} (\hat{q}_{res} - 1 + \sigma_{\hat{q}_{res}}^2) + \bar{N} (\bar{N} - 1) (\hat{q}_{qc} - 1 + \sigma_{\hat{q}_{qc}}^2) \} \sigma_{p_t}^2 \\
 & + \bar{N} (\bar{N} - 1) \sigma_{\bar{p}_t}^2 + \dots
 \end{aligned} \tag{53}$$

where $\hat{q}_{res} \in [1, 2+]$ represents the effective sibling number for secondary particles derived from a single primary parent (*e.g.*, resonance decays), $\hat{q}_{qc} \in [0, 2]$ represents the effective sibling number for two-point quantum correlations (HBT and FSI), $\sigma_{\hat{q}_{res}}^2$ represents event-wise fluctuations in \hat{q}_{res} , $\sigma_{\hat{q}_{qc}}^2$ relates similarly to \hat{q}_{qc} and $\sigma_{\bar{p}_t}^2$ represents event-wise variation of the parent mean (*e.g.*, distortion of the parent p_t distribution by event-wise fluctuating DCC or radial flow contributions). Thus, possible EbyE fluctuations in single-point *and* two-point aspects of the parent distribution are here included with possible sources of two-point correlation as contributions to A . Mean values of quadratic forms are essentially the only aspects of EbyE variation of the parent distribution accessible to a two-point or variance analysis. Note that an uncorrelated system ($\hat{q} \equiv 1$) and stationary parent ($\sigma^2 \equiv 0$) result in $A \rightarrow 0$, consistent with the CLT. This gives some impression of the complex task of interpreting a nonzero *or* zero value for A . Some unfolding of these possibly bipolar contributions can be achieved by noting multiplicity dependence, but there is not a unique inversion in general.

12.4 The subevent method

In the subevent method [19] each event is partitioned into two parts at large scale ($\approx L/2$), for instance by a pair of disjoint bins on rapidity, azimuth angle or a random particle index n . One then extracts the covariance of one or a pair of additive measures m ($= n(L/2), p(L/2) \dots$) or ratios $\langle m \rangle \equiv m_1/m_2$ obtained from the two parts. Effectively, the two-point density analyzed is $\rho(L/2, L; m_1, m_2)$ on G_2 . We can interpret this method within the present context by referring to Eq. (36). The LHS is the variance difference referred to in [19] as the ‘direct’ method (in contrast to the subevent method). For specific cases (scale intervals) the RHS is the same covariance which forms the basis for the subevent method. For these cases the quantity $A(\delta x_1, \delta x_2; x_1, x_2)$ can be identified with the quantity $\sigma_{x, dyn}^2$ in [19], thus also providing the relationship between Φ_{p_t} and the subevent method. (The discussion of Φ_{p_t} in [19] assumes lack of correlation between $\langle p_t \rangle$ and N , so the corresponding covariance term B in Eq. (52) is missing.)

One must carefully specify the scale interval being used for a given analysis. The subevent method as defined refers approximately to the scale interval $[L/2, L]$, whereas the ‘direct’ method as discussed in [19] and the Φ_{p_t} measure as it is usually presented refer to the interval $[a, L]$. The subevent method should therefore nominally be sensitive only to correlation developed or total variance accumulated in the interval $[L/2, L]$, and should not be directly comparable with an analysis based on $[a, L]$. However, because the partition methods employed are not applied directly to the primary space but only indirectly to a measure space (momentum space or a random index) it is possible for small-scale correlations to be included at some unknown level, prompting a recommendation to partition momentum space at large scale in such a way as to exclude issues of two-track resolution, BE/FD correlations and FSI. One effectively cuts on density $\rho(a, L; p_1, p_2)$ in $G_2(p_1, p_2)$ so as to exclude a neighborhood around the main diagonal (at small scale) which contains quantum correlations and final-state interactions.

Various multiplicity dependencies are considered in [19] as means to unravel multiple contributions to $\sigma_{x, dyn}^2$ similar to the contributions included in Eq. (53). Missing from [19] are possible contributions represented by terms σ_q^2 for resonances and quantum correlations from event-wise variation of the *two-point* parent distribution. As with all global-variables analyses direct analysis of two-point density ratios on space $G_2(m_1, m_2)$ provides a more complete way to unfold differing contributions to variance comparison measures when the data have sufficient statistical power to support this approach.

The subevent method is analogous to forming a mixed-pair reference from similar events, except that the ‘events’ compared are subevent components of a single event. Use of large-scale event partitions to establish a statistical reference may be difficult to interpret in some cases, albeit computationally convenient. Assumptions must be made about the correlation structure of events which may be difficult to justify. An alternative for the study of large-scale correlations may be to compare sibling *and mixed* subevents – subevent pairs formed in similar ways from the same event and from similar events. This procedure would form part of a general scale-local approach to two-point correlations spanning the scale interval from particle pairs to subevent pairs, with the same general analysis method used throughout.

12.5 Ratio fluctuations and resonance production

In [30] it is proposed to study event-wise fluctuations in hadron multiplicity ratios in order to determine resonance abundances at chemical freezeout. The method is based on a change in the fluctuation structure of $+/-$ multiplicity ratios to the extent that resonance decays (R) as well as thermal production (T) contribute to hadron yields. If a $+/-$ multiplicity ratio is defined by

$R_{12} = N_1/N_2$ and we assume no correlations among resonances ($\sigma_r^2 \approx \bar{N}_r$) then the relative variance in the ratio is given by (r is a resonance type index)

$$\frac{\sigma_{R_{12}}^2}{\bar{R}_{12}^2} = \left\{ \frac{g_1}{\bar{N}_1} + \frac{g_2}{\bar{N}_2} - 2 \frac{g_{12}}{\sqrt{\bar{N}_1 \bar{N}_2}} \right\} \quad (54)$$

where

$$\begin{aligned} g_1 &= \frac{1}{\bar{N}_1} \{ \sigma_{T,1}^2 + \sum_r \bar{N}_r \overline{n_{r,1}^2} \} \\ g_2 &= \frac{1}{\bar{N}_2} \{ \sigma_{T,2}^2 + \sum_r \bar{N}_r \overline{n_{r,2}^2} \} \\ g_{12} &= \frac{1}{\sqrt{\bar{N}_1 \bar{N}_2}} \{ \sigma_{T,12}^2 + \sum_r \bar{N}_r \overline{n_{r,1} n_{r,2}} \} \end{aligned} \quad (55)$$

If we further assume that $\bar{N}_1 \approx \bar{N}_2 \approx \bar{N} = \bar{N}_T + \bar{N}_R$, $\bar{N}_R \equiv \sum_r \bar{N}_r$, $\alpha_R \equiv \bar{N}_R/\bar{N}$, $\alpha_T \equiv 1 - \alpha_R$, $\sigma_{T,i}^2 \approx \bar{N}_T$, $\sigma_{T,12}^2 \approx 0$, $n_{ri} \equiv 1$, then we have $g_1 \approx g_2 \approx 1$ and $g_{12} \approx \alpha_R$. Therefore, $\sigma_{R_{12},obj}^2/\sigma_{R_{12},ref}^2 \approx 1 - \alpha_R$. These assumptions apply to correlation-free thermal pion emission combined with a single thermal resonance type decaying strictly to $\pi^+ - \pi^-$ pairs as a simple example.

Eq. (54) for $G_1(n_1, n_2)$ is a variant of Eq. (23) for $G_1(p, n)$ (equivalent in form if the latter is divided through by \bar{P}). There are two similar cases described by this algebra: distributions on $(n_1/n_2, n_1 \cdot n_2)$ and on $(n_1 - n_2, n_1 + n_2)$. The relative fluctuation distribution is essentially the same in the two spaces if a gaussian model and linearization are appropriate. The *obj/ref* ratio for multiplicity *products* (or sums) in the same case would be $\sigma_{\Pi_{12},obj}^2/\sigma_{\Pi_{12},ref}^2 \approx 1 + \alpha_R$ [16], where $\Pi_{12} \equiv N_1 \cdot N_2$. For the simple example considered, the thermal contribution is uncorrelated on $G_1(n_1, n_2)$, with equal variance on all variable combinations, whereas the resonance contribution is perfectly correlated along $N_1 = N_2$. A superposition of these two contributions with fractions α_T and α_R gives the stated result.

While correlations in the resonance contribution decrease the variance of the difference and ratio distributions they *increase* the variance of the marginal distributions on N_1 and N_2 and the sum and product distributions. The variance of $\langle p_t \rangle = P_t/N$ is also increased by the presence of resonances because the effective sample number N_{eff} is *reduced* by correlations. In the limit of all resonances decaying to pion pairs and no thermal pions the observed number of particles is twice the number of statistical *DoF* as reflected in an effective sibling index $\hat{q}_{res} \approx 2$ in Eq. (53), leading to a doubling of observed $\langle p_t \rangle$ variance over the CLT expectation for the observed particle multiplicity.

By some combination of fluctuation measurements of ratios/products or sums/differences and invocation of known resonance-decay schemes one can hope to infer the fractional resonance yield at chemical freezeout. Precision studies of resonance production, which are interesting in themselves and essential as a baseline to understand other aspects of the hadronic final state, require the formation of a reference system, so that one is dealing not with the ratio/product or sum/difference covariance system for real events alone but rather with differential departures from a reference. There is also the problem that other correlation sources may compete with resonance decay to alter N_+/N_- fluctuations.

12.6 Covariance analysis of global variables

In [31] it is proposed to increase sensitivity to possible QGP ‘signals’ by a combined analysis of the covariance of two or more global event properties. Event-wise antiproton multiplicity and transverse emitter radius are considered as an example. This approach is similar to the analysis format presented in Sec. 11.3, but without use of a reference to form a comparison measure. The centrality dependence of covariance is itself taken as the indicator of unexpected behavior. One interesting aspect of this proposed application is that one of the measures is not a simple distribution statistic like total multiplicity or mean p_t but a more complex correlation aspect of the two-point momentum distribution. This proposal is consistent with a general program to measure fluctuations of all kinds in the neighborhood of a phase boundary in order to better understand the underlying Lagrangian by enumerating and identifying the relevant *DoF* for various constraints.

The underlying model premise in [31] is that, depending on centrality, collisions may fluctuate between two disconnected event classes, one (q) manifesting characteristics associated with QGP formation (*e.g.*, increased emitter radius and antiproton abundance mean values) and the other (h) manifesting conventional hadronic systematic variation with participant number. The measure system in this case is $(m_1, m_2) = (N_{\bar{p}}, R_t)$, and the centrality dependence of the covariance is contrasted with that of the mean values of the measures for sensitivity in terms of this fluctuation model. While the paper emphasizes covariance measures it admits that variances would exhibit similar centrality dependence for this model. The central point is that attention to fluctuation measures is critical for detection of intermittent phenomena in collisions.

The basic elements of the simulation are the estimated baseline measure fluctuations in the hadronic and QGP regimes, and the assumed shifts in mean values from one event class to the other. These assumptions determine all relevant aspects of the simulation. Mean values for the two event classes [h,q] are given by $\bar{N}_{\bar{p}}(b) \approx [56, 78] (N_{ch}(b)/N_{max})$ (antiproton multiplicity into STAR acceptance) and $\bar{R}_t(b) \approx [6, 9 \text{ fm}] (N_{ch}(b)/N_{max})^{1/2}$, where N_{ch} is the charged-particle total multiplicity used to infer event centrality and N_{max} is the mean value for central collisions. Variances are approximated by $\sigma_{N_{\bar{p}}}^2 \approx \bar{N}_{\bar{p}}$ and $\sigma_{R_t}^2 \approx \bar{R}_t^2$, the second dominated by an estimate of measurement error rather than intrinsic statistical fluctuations. The result is a rather large increase in covariance in a crossover region of centrality where the two event types are nearly equiprobable. We observe in a scatter plot on $(N_{\bar{p}}, R_t)$ that the joint distribution is bimodal due to the chosen relationship between assumed mean-value shifts and variances.

The assumption of catastrophic fluctuations between two event types maximizes the effect of the model on variances and covariance. An alternative scenario consisting of a partial transition (still first order) over some fraction of the event volume (having its own fluctuation structure) would produce a significantly smaller effect on correlation measures. These model assumptions do however correspond to some popular QGP scenarios. We note that an expected major source of covariance, collision geometry and stopping fluctuations (‘volume’ fluctuations) as illustrated in Fig. 7 and present for either event type, is here probably underestimated by the Poisson variance of N_{ch} .

Having modelled the most dramatic QGP scenarios one might then consider how to establish by these correlation measures either that a QGP hypothesis is falsified at some level, or that some fraction of some events does indeed undergo a transition to a nonhadronic phase. In the adopted scenario the input to the correlation measures is bimodal, a rather strong statistical manifestation. A weaker and possibly more realistic situation to simulate is one in which anomalous events form a tail structure on a hadronic core. In this case one has to compare carefully the centrality dependence of covariance matrix elements to that for a reference and perform a significance study of the result.

There is much technical work needed to elaborate these and other EbyE analysis methods. We see from these examples and interpretations that a general description of scale-local covariances and comparison measures permits a comprehensive interpretation of some apparently dissimilar methods.

13 Toward a Generalized Central Limit Theorem

From the original concept of the elementary central limit theorem as an integral variance comparison measure for a gaussian fluctuation model, we have enlarged the analysis to multiple measures described by a covariance matrix, have expressed the basic comparison in a scale-dependent manner, have expressed the source of departure from the CLT in terms of a matrix of scale integrals comparing object and reference two-point densities and have considered as the most differential form of comparison a direct analysis of density ratios on two-point measure spaces.

From the vantage point of this scale-local picture we can now generalize the central limit theorem and reinterpret its role in the description of many-body systems. In terms of the new concepts the elementary CLT as presented in Sec. 6 compares total variance for a single measure at two points on scale to determine whether additional variance has accumulated over the interval. The CLT states that if there is no net correlation introduced within a scale interval the total variance of a measure is the same at each endpoint – $\Sigma^2(p, p, \delta x_2) - \Sigma^2(p, p, \delta x_1) = 0$.

We have subsequently developed a greatly extended version of this elementary comparison procedure and generalized it to a pair-wise total (co)variance comparison for an arbitrary measure system as represented by

$$\mathcal{K}(\vec{m}, \delta x_2) - \mathcal{K}(\vec{m}, \delta x_1) = \mathcal{A}(\delta x_1, \delta x_2; \vec{m} \otimes \vec{m}) \quad (56)$$

and have given the general form of the source of any total variance change in terms of a matrix of covariance integrals over two-point measure densities represented by \mathcal{A} . We can therefore generalize the CLT to a statement about the scale invariance of covariance:

The total covariance matrix $\mathcal{K}(\vec{m}, \delta x)$ for a measure system on the space Ω is *scale invariant* over a bounded scale interval $[\delta x_1, \delta x_2]$ in the absence of all categories of *net* two-point measure correlation over the same scale interval. Departure from scale invariance is represented by covariance integrals of density differences on corresponding two-point measure spaces.

In this formulation we have represented the possible sources of accumulated covariance as integrals over two-point measure spaces. We have also introduced the possibility that a uniform reference (implicit in standard variance calculations) may be replaced by an arbitrary reference or model appropriate to a given problem. The resulting system offers the opportunity to choose from a continuum of techniques and references, depending on the fluctuation model and statistical power of the data.

14 Conclusions

We have as our nominal goal identification and study of a quark-gluon plasma, a simplistic terminology for a complex phenomenon. It may be that the emergence of a color-deconfined QCD system is subtle and topological rather than catastrophic and thermodynamic. Production of a QCD plasma as a catastrophic process with discontinuous signals may not be forthcoming. To

insure the best opportunity for discovery we require sensitive quantitative comparison measures which extract complete correlation information from collision events.

In studying the correlation structure of HI collision events at the CERN SPS we have been struck by their sameness and ordinariness, the apparent close adherence to simple statistical fluctuations in the distribution of mean p_t and the wide-ranging description of hadronic ratios and abundances by a simple statistical model with a universal chemical freezeout temperature. These two results are deeply connected by their relationship to the central limit theorem and related maximum symmetry principle. A common supposition has been promoted that QCD plasma formation at high energy densities may lead to dramatic, or easily quantifiable, changes in the correlation structure of the multiparticle final state, changes detectable by elementary correlation measures. Given the unspectacular SPS results we are now forced to re-evaluate our need for sensitivity.

A central goal in EbyE analysis has been to develop global comparison measures sensitive to excess variance which might signal residual correlations due to incomplete equilibration – possibly structure remaining from a phase transition. We have seen that these global measures are ultimately based on the central limit theorem as a statistical reference, and the CLT is in turn based on a scale-integral concept. The present treatment makes it clear that global-variables analysis based on scale integrals – an unavoidable strategy in case of low statistical power – is rather limited in its sensitivity, interpretability and power to discriminate among various correlation sources.

Statistical power (event multiplicity) determines the optimum scale interval and scale resolution of a correlation analysis. Low statistical power requires the use of integrals over space and scale to extract statistically meaningful correlation measures, with accompanying loss of information. Higher statistical power makes possible extension of the analysis to scale-local differential measures and direct analysis of densities on two-point measure spaces. One can then study the scale dependence of density ratios to extract full correlation information from data. Future developments include direct determination of two-point densities, analysis of scale-local topological measures extracted from these densities and extension of the analysis to higher moments ($q > 2$).

The question of scale invariance represented by the CLT is deeply related to the nature of distribution symmetries. Scale-invariance is equivalent to constant autocorrelation – Eq. (38) – which is in turn an indication of maximum symmetry. Thus, the CLT in its generalized form is a statement about symmetry variation over a bounded scale interval, *a potentially new insight*. With a uniform distribution as reference one can inquire whether an object distribution is maximally symmetric over a scale interval and how its symmetries evolve dynamically. More generally, one can ask whether object and reference distributions have the *same* symmetry or autocorrelation over some scale interval. This presents a new approach to quantitative description of nonequilibrium systems and the equilibration process.

Scale-dependence of symmetry variation is the underlying issue in the study of phase transitions. Near the QCD phase boundary hadrons and partons as dynamical objects are transitory and ill defined. Basing a dynamical description on a particle hypothesis near a phase boundary is a poor strategy. To provide a fundamental treatment one must ultimately adopt a descriptive system in which particles are not fundamental elements, in which arbitrarily correlated measure systems play the central role. A generalized central limit theorem provides a basis for such an approach. One can study the symmetry dynamics of a measure system relative to a maximally symmetric system or other symmetry model.

In a broader sense we are probing the dynamics of vacuum symmetries within a violent collision. The more traditional study of symmetry *statics* is carried out in asymptotic regions of a constraint space far from any phase boundary. Symmetry *dynamics* is important in the neighborhood of a phase boundary. The QCD phase boundary marks a region of constraint space where the nature of

vacuum symmetries is changing dramatically and fundamentally. This is a challenging undertaking requiring the best possible analysis techniques.

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